Mathematics Teacher

DEVOTED TO THE INTERESTS OF MATHEMATICS IN JUNIOR AND SENIOR HIGH SCHOOLS

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Should We Teach Statistics in the Senior High School?
GODFREY H. THOMSON 129

A Better Use of Tests in Mathematics W. D. REEVE 140

New Types of Schools in Germany and Their Curricula in Mathematics DR. W. LIETZMANN 148

A Number Play in Three Acts 154

Purpose, Method and Mode of Demonstrative Geometry 170

The Teaching of Logarithms R. C. CHRISTOFFERSON 178

New Books 189

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THE MATHEMATICS TEACHER

VOLUME XVII MARCH, 1924 NUMBER 3

SHOULD WE TEACH STATISTICS IN THE SENIOR HIGH SCHOOL?[1]

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Strictly one ought to ask, should we teach *more* statistics in the senior high school; for already we teach some. All pupils know what an average is, are accustomed to drawing graphs in lessons on rainfall, on imports and exports, on social conditions, and the like. The chief reason why I shall urge that the teaching of statistics should be more conscious and should be more complete is, that an understanding of their interpretation is more and more needed in everyday life, in newspaper reading, in comprehending the appeals of this or that political party. And as a subsidiary reason I might add that statistical ideas form links binding together many other subjects as far apart as biology and physics, insurance, and engineering; and I attach great importance to anything which helps to integrate into one unity the intellectual life of the high school, and to prevent it from becoming so many water-tight compartments.

But although statistics are quoted freely in all discussions of affairs of the day, they seldom have a plain tale to tell; life is too complex for that. And so it comes about that the same facts are quoted, with slight change of emphasis, by radically opposed parties, and the man on the street may be forgiven for repeating the gibe that "anything can be proved by statistics," or that other epigram about "lies, damned lies, and statistics." Some twelve months ago in England a prohibition campaign was in full swing, and the hoardings—the billboards—were placarded with statistics showing the decrease of crime in this country after the introduction of the Volstead Law, the emptying of the workhouses, and similar points. But the opposing party put up what were on the face of them equally convincing arguments and figures also taken from this country, showing

[1] An address to the New York Section of the Association of Mathematics Teachers, January 18, 1924.

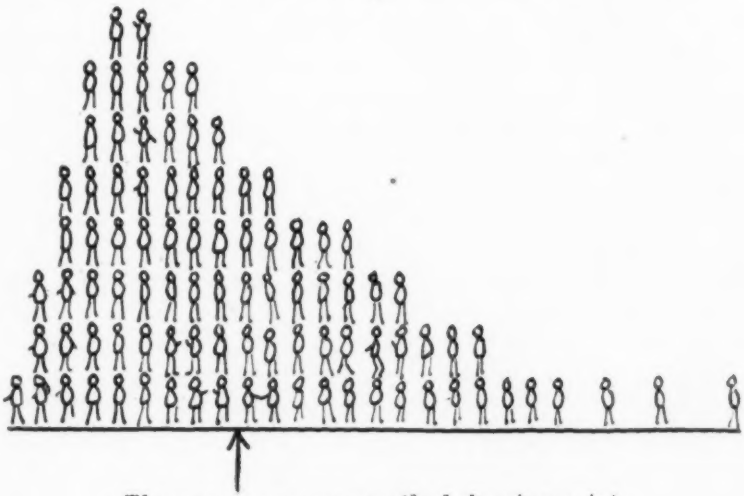
increase of crime due to bootlegging, increase of drug taking, and I forget what else. Or turn to an illustration from Italy, where between 1885 and 1912 the savings index rose steadily from 47 to 192, and the index of consumption of tobacco from 82 to 151, showing apparently that consuming tobacco is an aid to savings. A dose of common sense would guard one against accepting such a conclusion, but would hardly enable one to examine the statistics with a view to deducing the real correlation, between tobacco-smoking and saving; and in another instance absurdity might not be so obvious. Even a strict application of statistical procedure may, without the saving grace of an understanding of the fundamental laws, lead to fallacy, just as is the case with logic, in which connection I am reminded of the story of the Oxford under-graduate who got drunk on Monday on whiskey and soda, on Tuesday on brandy and soda, and on Wednesday on gin and soda; after which he swore off soda for life.

Considering then the part which statistics play in the political and social life of a democracy, there seems a real need for educating in their use such a proportion of the electorate as have the necessary *I.Q;* and such are found in the high schools.

Time does not permit one to speak of the more visual treatment of statistics, such as the construction of diagrams, curves and maps, the most suitable scales and methods to employ for those purposes, and the virtues and vices of different ways of tabulating numerical data. Among others, the teacher of geography usually is, or ought to be, expert in this province, and knows to what sources to turn in libraries and elsewhere in order to obtain the most up-to-date facts. I would like with your permission rather to direct attention to the more theoretical and mathematical side of the subject, and first of all indicate what I think is the minimum equipment of technical concepts and terms, and how these can best be taught. The first of these points is the way in which measurements of any fact of human life tend to be distributed in a certain manner about a central measure, of which the best known is the "mean" or "average". In leading up to this, it is a good plan to take some quantitative fact actually appertaining to man (such as say size of income) and draw on the blackboard a number of men standing at their appropriate places on a horizontal scale. Then, since some of

the men will have the same income, draw them standing vertically on each others shoulders as shown in the next figure.

In such a distribution there will be individuals at each end, as a rule, who are all alone, with no comrade; the richest and the poorest (though I fear there is some "jamming" at the poor end); the tallest and shortest; the highest and the lowest intelligence. But as we near the middle of the distribution the more ordinary men are standing two, three, perhaps hundreds deep, the whole forming a curve of a type which a student of statistics comes to be very familiar with. And from such "dis-



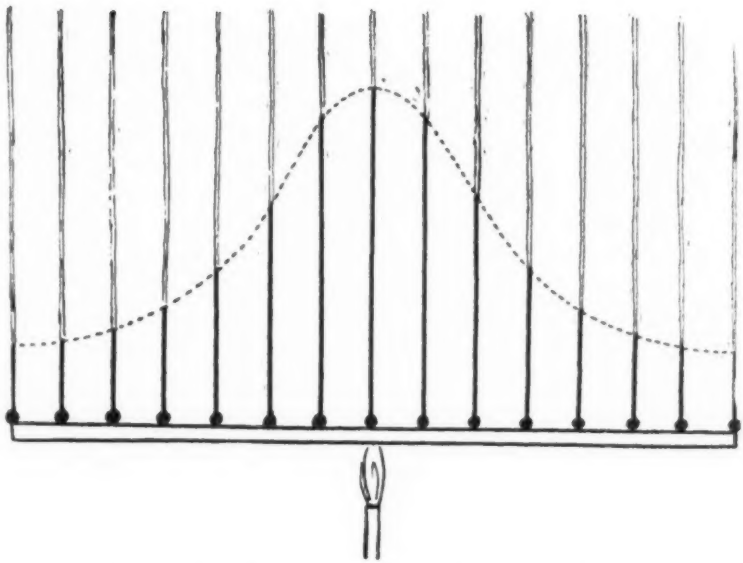
The mean or average as the balancing point

tribution curves" we learn one of the most important facts of life, namely that (at any rate in things of Nature such as height, or intelligence, however it may be in incomes) there is only one peak to the curve, only one type, about which there are individual differences of startling amount and frequency. The pupils will already know how to calculate an "average". But it will be illuminating to realize that, if we suppose the men as above distributed to be standing on a plank and to be each of the same weight then the average is the point about which the see-saw would balance, shown by the arrow. If the distribution is "skew", as we say, there will not be so many men on one side of the balancing point as on the other, but those who are fewer

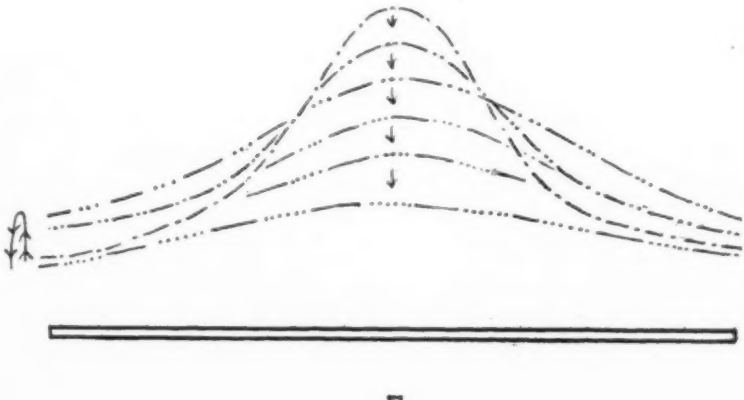
make up by having some of their number a long way along the plank, exerting a more considerable leverage. This leads on to the consideration of the "median"—the income possessed by the man who has half the crowd to right and half to left—a man in this particular hypothetical diagram standing somewhere on the left of the arrow which marks the average. For some statistical purposes the median is a better measure of central tendency than the average. Consider for example a jury charged with the task of assessing the damages to be paid by some accused person, and let us suppose they agree to write down, each on a piece of paper, the sum they consider just. To take the "average" of such sums would put it in the power of any vindictive juryman to make the damages as big as he pleased, by setting down a large enough sum on his ballot paper, whereas such a proceeding on his part would leave the "median" unchanged (I believe I borrow the instance from Sir Francis Galton but my memory is not confident).

Similarly the "mode" or most frequent value of income will in a skew curve differ from both average and median. For a man selling shoes it is more important to know the "mode", the fashion, the most frequently worn size, than the average or the median. In symmetrical distributions, as the see-saw illustration at once makes obvious, the mean mode and median coincide. The symmetrical distribution curve, a special form of which is known as the "normal probability curve", is particularly interesting from the way in which it or some similar form comes up in many departments of science.

For example in the conduction of heat outwards along a plate of metal to the middle of which a flame is applied, thermometers at regular intervals would show that the curve of temperature has such a distribution form. The energy is passed along by a molecular mechanism of some sort which depends on the law of chance, each molecule passing on energy to its neighbors at random, as one might pass pamphlets from hand to hand in a crowd. If in the middle of such a jostling throng an individual with a supply of pamphlets started to give them away at random to individuals within arms length, and each individual who had a pamphlet gave it at random to someone else, or rather halved it with him, sometimes passing in-wards sometimes outwards, there would be a steady spread of the paper, and seen from



Spread of heat outwards in the metal plate



After ceasing to heat the plate

above the crowd would show a centre of "paperiness" at the distributor (the source of heat) and gradually diminishing "paperiness" at a distance therefrom. If the source of the pamphlets became exhausted, but still everybody had a perfectly random impulse to choose a neighbor at a venture and equalize pamphlets (so that if he had four and I two I would take one) then by the operation of such a chance law the pamphlets would gradually spread themselves over the crowd equally. And if we add the idea that pamphlets from time to time get trampled underfoot, so that ultimately the crowd returns to its paperless condition, the analogy with the heating experiment would be very close. There, when we cease to heat, the curve of temperature would cease to rise. It would then suffer a general fall by the general cooling of the plate: and a change of shape owing to the fact that heat would still pass, on the average, from the hotter centre to the cooler edges. The heights of the thermometers would pass through a series of changes like those shown on the preceding diagram by the dotted curves representing a series of distribution curves of increasing "deviation" or "scatter". Teachers of physics will find the germ of this illustration in one of the middle pages of Clerk Maxwell's "Theory of Heat"; and teachers of mathematics will find (strictly for their own consumption) the equation of the "normal curve of distribution" deduced from Fourier's differential equation for the conduction of heat by the late Lord Rayleigh in the "Philosophical Magazine" of February, 1899.

In pure mathematics the binomial theorem has the most obvious connection with the law of chance, *is* indeed the law of chance. It can be used with advantage to illustrate chances of obtaining so many heads in so many coins, and in similar experiments, and can even be taught in that way. The binomial coefficients

$$\begin{array}{ccccccc} & & & 1 & 2 & 1 & \\ & & 1 & 3 & 3 & 1 & \\ & 1 & 4 & 6 & 4 & 1 & \\ 1 & 5 & 10 & 10 & 5 & 1 & \end{array}$$

pass over in the limit into our friend the normal law of distribution.

Any coin tossing or other chance experiments made in the mathematical laboratory (and I would like to see every high

school equipped with a mathematical laboratory, as distinct from the equally valuable physics laboratory) could, if properly planned, be used right away by the teacher of biology in explaining simple Mendelian laws, where pollen and ovules with or without a certain Mendelian unit factor take the place of the coins which fall head or tail.

In engineering shops, illustrations of statistical laws can be seen by those who have eyes for them. One of the best examples of a "skew" curve I ever gathered was from the coefficients of elasticity of a number of samples of steel. Then again in mass production the margin of error allowable is important, and leads one at once into the question of how errors add together. If a cogwheel, say, is to be produced in large numbers for a certain place in a machine, then its breadth from face to face will be specified by limits within which each specimen must lie, or it will not go into its place. But although all the specimens lie within these limits, they will not as a rule be scattered equally frequently over the permissible range. And if a number of cogwheels have to be mounted on the same spindle, then the limits which are specified must take into account the possibility of a number of cogwheels, each of the upper limit of thickness, happening by chance to come together on to one spindle. The probability of this must be calculated, and boys interested in engineering will have more patience with the binomial theorem and with square root in algebra if they know that these abstract matters are necessary for those who would plan mass-production operations.

A similar knowledge of the laws of addition of errors (they add according to a square root law) is found in surveying. If a surveyor makes 100 observations which are to be taken one after another, then errors will accumulate; but they will sometimes cancel, and the net result is that instead of the accumulated error after 100 readings being 100 times as great, it is only 10 times as great.

The binomial coefficients spoken of above give the relative chances of certain events happening. For example the numbers 1, 4, 6, 4, 1 out of sixteen represents the chance, in sixteen throws each of four coins, that we obtain

0 1 2 3 or 4 heads.

In the long run, a throw of 4 coins is likely to be all heads in one sixteenth of the throws; to be 2 heads and 2 tails in six sixteenths of the throws. Just as here there is a most probable arrangement (2 heads) occurring frequently, and less probable arrangements occurring infrequently, so, for example, the atoms of soft iron in a bar have, in their various groups, more and less probable arrangements. When a magnetic field is slowly applied to the bar, and slowly strengthened, these random arrangements of atoms yield one after another to its pull, and straighten themselves out—the bar becomes slowly magnetised. Unstable but rare groupings will yield first, then the larger numbers of more stable arrangements, finally the very closely locked groups, which again are rare. To use the same numbers as we obtained in our coin throws above (to clinch the illustration), as our magnetic field increased through strengths represented by 0, 1, 2, 3, 4 units the groups of iron would straighten out in the proportions 1, 4, 6, 4, 1 out of 16. The magnetisation of the bar at any point would be the number of groups of atoms straightened out up to and including that point, in

Field 0	1 group	1
" 1	4 + 1 =	5
" 2	6 + 5 =	11
" 3	4 + 11 =	15
" 4	1 + 15 =	16

The numbers 0, 1, 5, 11, 15, 16, if set out as a curve over the strength of field, show the characteristic shape of a "hysteresis" curve, which is in fact the integral of a probability curve, or something closely resembling it.

Curves very similar to (though not perhaps identical with) this "probability integral" are also obtained in radioactivity experiments. When certain substances are removed from a radioactive mixture, they sometimes show the phenomenon of having no radioactive qualities at first, but then gradually recovering them. This recovery shows the properties of a curve dependent upon chance. The explosion of an atom which is the basic occurrence in radioactivity is a rare event dependent upon a coincidence of circumstances, as 50 heads in 50 throws is a rare event. For analogy with the radioactive recovery, we may compare the radioactivity of an element with the birthrate of

a community. Were all the children removed from the community and placed upon an island, this island colony would at first have no birthrate (no radioactivity) but it would gradually show births, slowly at first, then in a large increase, then the rate would gradually go up to its steady state, with the same kind of inflections as the above numbers 0, 1, 5, 11, 15, 16.

This instance or analogy reminds us at once of another province in which statistics are all important, namely vital statistics and insurance. Here some knowledge of the operation of insurance principles is an important item in the equipment of the future citizen. A not dissimilar branch of statistics is that which of late years has grown up round the subject of intelligence tests. In this realm, a proper understanding of the nature of distribution curves leads to a better appreciation of one's fellows. For a bald statement of the difference in average performance in intelligence tests between white and negro, or Nordic and Mediterranean immigrant, or officer and private, may be most misleading, whereas a comparison of distribution curves would show a tremendous overlap, and prove that individual differences are far more important than any other; an appreciation of which might go some way towards minimizing racial and social prejudices.

The idea of a coefficient of correlation is one which has become increasingly common in the jargon of experimental psychology and pedagogy of recent years; and indeed its introduction by Sir Francis Galton in the nineties of the last century was an addition to the armoury of logical weapons which has had important and far reaching consequences. In its more elementary guizes it is readily enough understood by non-mathematical laymen. A correlation coefficient is simply a fraction which enables us to predict how much of a quantity x may be expected when a quantity y is observed. For instance, the correlation between father's and son's height enables us to predict the most likely height of a given man's son. If the correlation is not very high the predicted height, although the most likely, may not be very likely; just as 50 heads in 100 coins, though the most likely event, is not very likely. Pondering over such an apparent paradox is not an altogether unprofitable mental exercise.

The easiest form of calculation of a correlation coefficient—it is not an accurate way to do it but serves well as an introduction—is to employ a device which we owe to Mr. Udny Yule. By its means a geography class can correlate such things as sunshine and wheat production, east and west winds and rainfall, and the like. For a hypothetical example take the correlation between Latin and English marks in a class. A perfect correlation would be indicated by $r = +1$, in which case every child would have the same relative position in each subject. If there were no possibility of predicting English standing from Latin standing, if the lists showed not the slightest resemblance to one another, the correlation would be zero. We want to be able to calculate a fraction which will quantitatively express the resemblance in intermediate cases; and the approximate method referred to above is shown in this example. 11 pupils (let us suppose) are above the class average in both Latin and English. 9 are below the class average in each subject. 6 are above in Latin but below in English. 4 are on the contrary better than average in English but worse in Latin. Then the fraction

$$\frac{\sqrt{11 \times 9} - \sqrt{6 \times 4}}{\sqrt{11 \times 9} + \sqrt{6 \times 4}} = 0.3$$

will be found to be a fair approximation to more correctly calculated values. The next step would be to use a printed correlation blank like that of Dr. Otis, where there are many compartments instead of only four, and where the calculation required is guided by the lettering on the blank. Such blanks could then be used for sociological or geographical data. I think that the calculation of correlation coefficients is a case where empirical familiarity with the method has to precede an understanding of the *raison d'etre* of the details of the calculation. Later, the calculation can first be made plausible and then in some cases can be furnished with a theoretical background.

The title under which I have been speaking tonight appears on your program as:—"Should statistics be an elective subject in the senior high school". I have said nothing whatever about the "elective subject" part of this title, because I know so little about your elective system. But in reply to a question which has just been addressed to me by Miss Sanford, I should per-

haps say that my instinctive reaction is against making a separate subject of statistics, or indeed of geometry or trigonometry or any other branch of mathematics. I like them well mixed up; and my suggestions tonight have been directed towards ways in which the theory of probability can be seen to permeate other subjects, rather than towards making it a subject. But if a mathematics teacher happens to be enthusiastic about it, by all means let him have a little course in addition to the occasional pickings.

A BETTER USE OF TESTS IN MATHEMATICS[1]

By W. D. REEVE
Teacher's College

Tests in mathematics may be classified under two main heads, namely, prognostic tests, and achievement or diagnostic tests. Prognostic tests are those which may be given by the teacher early in the pupil's mathematical career the results of which are supposed to measure innate ability to do mathematical work. By their use one is supposed to be able to predict how the pupil will probably perform in his later mathematical work. These tests must, in the nature of things, be more comprehensive than any which have been devised for particular fields.

The necessity for such tests grew partly out of the fact that most of the previous tests in mathematics dealt only with the mechanical phases of the work. Accordingly, they were objected to by many professional mathematicians who asserted that there is little in common between the habits of symbol manipulation which practically all of the early tests emphasized and the more fundamental processes so characteristic of higher mathematical study.

In this way the professional mathematicians pointed out that the intensive tests in particular fields like arithmetic, algebra, and geometry were not what they should be inasmuch as none of these subjects are applied in isolation. The validity of this statement would of course depend upon the purpose the makers of the particular tests had in mind. No one to my knowledge has claimed that any prognostic test can be made the absolute basis for prediction and yet they constitute a step in the direction of trying to find out the pupils who are the most probable mathematicians in the school. If they can be of genuine assistance in helping us in our problem of discovering individual differences in ability, they can be put to good use in any school. We should insist, however, that they are not to be used as an absolute basis. Absolute dependence upon such tests means that the human element will be overlooked.

Speaking of Dr. Rogers' prognostic tests, Dr. Thorndyke says[2]: "Dr Rogers' tests are the best so far published to select

[1] Read before the National Council of Teachers of Mathematics, Chicago, Feb. 23, 1924.
[2] The Psychology of Algebra. E. L. Thorndyke.

according to promise of ability in algebra and geometry. We have evidence, however, that their value consists chiefly in their being a good measure of abstract intellect, and that any reliable measure of abstract intellect would prophesy success in algebra and geometry nearly as well. Also it seems probable that algebraic ability and geometrical ability differ nearly if not quite as much as do ability in algebra and ability in any other abstract subject such as physics or Latin. Consequently tests specialized for numerical and spatial data may be found to do this prognostic work even better than the Rogers tests.

If these tests could be arranged so as to be given before any algebra or geometry had been studied, it would of course be an added advantage. The Rogers tests presuppose a few month's study of algebra, and are much influenced by familiarity with the idea of procedure in geometrical proofs.

It seems probable that such a prognostic test for algebra could be made by using the Rogers tests 3 and 5 with a test in arithmetical problems and a test in disarranged numerical equations."

Probably the best work in connection with the points discussed above is that of Kelly which along with others is given in a very complete survey of tests prepared by Professor C. B. Upton and published as Chapter 13 of the National Committee report on the Reorganization of Mathematics.

If such tests can be made reliable as a help in prognosis, then I would favor a wider use of them in determining those who have innate mathematical ability. Moreover, I would then favor requiring mathematics beyond the Junior High School only of those pupils who come up to a certain standard requirement. That is, I should favor making mathematics beyond the years of the Junior High School for the most part elective. A better use of these prognostic tests would then come in our use of them in doing justice to a large number of pupils who are forced to study mathematics against their will and in many cases without profit.

Teachers are everywhere more familiar with the diagnostic tests. In fact it is more nearly what they make for their own pupils and what they as pupils were accustomed to take. They are intended to measure the progress which a pupil or a class is making in a given topic. To be sure they may also measure

innate ability since whatever progress a pupil makes is necessarily dependent upon the natural ability of the pupil. However, it is clear that these tests are affected by classroom influences to such an extent that they cannot be accepted as reliable instruments for measuring the innate ability of the pupils. Often, it is certain, they test results of formal drill work where often little if any real thinking is done.

For the purposes of this discussion we may divide diagnostic tests into two kinds. First those set by outside bodies often having little or no contact with the instruction that is given, and second, those set by some one who is responsible for the course of study that the classroom teacher is using. This may be the teacher himself, the teacher together with his colleagues, or someone familiar with the specific course that is being taught.

Of all of the tests set by outside bodies those made out by the College Entrance Board, certain state examinations, and the so called standardized tests are well known examples. These tests are good to the extent to which they test what ought to be taught or perhaps what an individual teacher or group of teachers think ought to be taught in a particular course.

Such tests have certain redeeming features among which I may mention one or two. It is often difficult for a teacher to drill his pupils definitely to pass a particular test of this kind. Moreover, the people who formulate standardized tests are often better able to judge what is fundamental in a course than are some teachers of mathematics. As a result their tests should be better than those of a teacher who may be more or less incompetent. On the other hand they have certain very obvious disadvantages. Such tests are often unfair and in a true sense not diagnostic because they do not test what is being taught in many schools. One could conceive of a situation where a class or most of a class in algebra might make a poor showing on a test set by an outside body and still be able to pass a rather rigorous test in algebra which is based upon the type of material they had been studying.

Another difficulty with such tests is that often they contain material that is not only unfair, but material that has already been discarded by progressive teachers. From a modern view point such material cannot be justified and yet if it is placed in tests that are widely given, it places undue emphasis upon

material that should not be taught at all much less be included in any list of fundamental questions. The trouble is that, if left in these tests, many teachers will get the wrong idea of their importance and will over emphasize them in their teaching. In a recent article[1] Dr. David Eugene Smith points out in more detail some of the dangers referred to above. For example, he quotes from a recent test among others the following problem:

$$\text{Divide } \frac{\sqrt{2}}{\sqrt{3}} \text{ by } \frac{\sqrt{2}}{3}$$

He then adds, "Why should anyone ever wish to perform this division? It is not algebraic, and it has no particular significance either in algebra or in connection with formulas that are likely to arise. It is possible to test our schools on something that we have some other reason to perpetuate than the mere reason for meeting this kind of test." One can find the same type of problem in almost any test he picks up where the test has been made up by one who is not careful in his thinking about relative values of content material or one who is not familiar with what is actually being taught. This may be due largely to the failure of the text book writers and the makers of tests to realize that the material of the present day should be different from what it was twenty years ago on account of the different character of our high school population. Our high school population is today different from that of twenty years ago "not only in their experiences and their interests but also in their inborn abilities."

Mr. Harl R. Douglass in a recent article[2] has pointed out shortcomings of our test in Algebra and states that he has attempted "to devise and standardize two series of tests to measure achievement in first year algebra and to afford a rough means of diagnosis." These tests or any others will be satisfactory to the degree to which they test what teachers of algebra think ought to be taught.

Thus, teachers are beginning to see that a test which merely bears a test label is not necessarily one to use. If it tests what

[1] On Improving Algebra Tests.—David Eugene Smith. Teachers College Record. May, 1923.

[2] Measuring Achievement in First Year Algebra.—Harl R. Douglas. The Mathematics Teacher. November, 1923.

ought to be taught and what is taught it will be considered a good test, otherwise it will not secure their approval.

It is therefore not the standardized test or any other test as such that I am worried about, but the wrong use and interpretation of such tests. The testing movement is one that should be encouraged, but there is room for improvement in the general administration of such tests.

May I venture to suggest that the new type of test is growing out of the feeling that teachers themselves, or at least those who are responsible for what the teachers are doing, should make the tests. The plan is to have the course of study in mathematics agreed upon first. Having done this, our next problem would be to list the minimum essentials and let them guide us in listing fundamental test questions. Thus, nothing would be included in any test which did not represent something worth perpetuating.

Such tests will be truly diagnostic in that they will aid in future instruction. Too often tests have been given purely for the sake of testing or at best as a means of securing marks. Unless tests can be used principally as a basis for improving instruction then it is not worth while to give them at all.

Let us suppose for example that we are all agreed that somewhere in the course we shall teach simple equations and that we shall teach only those of a certain type of difficulty—the final difficulty, however, to be determined by what the children can actually do. Let us agree that we shall try to test two things, namely, the ability to solve certain types of equations and the ability to translate certain word statements into algebraic equations. It is obvious that only the writing of the equation is necessary in the second case since the pupil's ability to solve the equation has already been tested by a test of the first kind.

Let us further agree that $X+5=9$ represents the first type of equation presented and that we want the pupil to learn to be rapid and accurate in solving this type of equation having first learned that to solve such an equation he must subtract the same number from each side of the equation. When the teacher feels that it is time for a test on this type he may give from twenty to thirty examples. Probably it might be a good thing

to give this list as a timed test, say two or three minutes, the time to be fixed so that only one or two of the pupils can finish it in the given time. It should be pointed out, however, that accuracy is the main thing.

Further, let us suppose that other tests represented by the types X—3—8=, 2X=10, X2=5, are given to test the pupils on the other three fundamental processes used in solving equations and so on through the work on simple equations. This will test their ability to do the formal work both with regard to speed and accuracy. The same thing can be done with a list of applied problems for which they are asked to write the equations. The same procedure can be carried on throughout the course.

A study of the results of these short time tests will reveal many valuable things to the teacher. If he pools his results he can develop norms for future reference and stimulate his classes to improve the standard.

One of the unfortunate results of established norms is that often a teacher is satisfied if he succeeds in getting his pupils to a certain district or national standard forgetting entirely that with better teaching everywhere the norm might be improved.

At the end of the work on simple equations there should be a composite test made up of all the various types of equations considered essential in the course and enough similar exercises added to make the test long enough to keep the brightest ones busy till the poorest ones have covered the minimum essentials included in the test. I have found it better to give this composite test at a time when the pupils could have time to finish at least the fundamental and essential parts. The main idea here is to find out whether, after all, the pupils can solve all of the different types. If some of the pupils cannot solve all of the exercises, we want to know which exercises they are and, if possible, why they could not be solved. It may be that the teacher will discover that his teaching has been inadequate in which case the teaching must be improved.

Many interesting and helpful things will grow out of such a procedure among which I may mention the following:

1. By checking his results the teacher has a measure of the relative values of the performance of his class. If marks must

be given, this furnishes a better basis than the traditional method.

2. If the results are charted with the names of the pupils at the left, the numbers of the exercises at the top, and checks for right answers and crosses for wrong answers in the body of the chart, the teacher can see at a glance where the trouble has been and can then begin to introduce remedial work.

3. Not all parts of a subject, of a topic or even the exercises in the same simple list are equally difficult. This discovery is still to be made by some people who mark test papers.

4. Pupils often fail to solve an exercise not because the exercise was intrinsically too hard, or because they have been lazy, but because the teaching has not been adequate.

5. Problems may be easy enough from an algebra standpoint but too difficult for the pupil as a problem in reading.

6. Certain traditional topics that have been thought too difficult for a certain grade are easily understood if the teacher presents the topic carefully. Conversely, some things that the teacher thinks are simple prove to be very difficult for certain pupils.

7. Few teachers anywhere have any clear notion as to how long it takes to teach anything to secure a satisfactory degree of mastery.

These matters can be settled by a better use of tests. For example, we ought to be able to find out how long it takes to teach a certain group of pupils to factor "the difference of two squares" type to a degree of mastery upon which agreement can be made.

8. The right kind of thinking about some of the formal processes in algebra may be conducive to rational thinking as well as problem solving.

The significance of such a program of testing is evident. In the first place, the teacher knows that he must be more alert in his teaching and that he must be particularly keen as to the salient points in the course. He must give tests that will help him to diagnose his own teaching situation. If he learns that his teaching has been poor, then he knows that he must be more careful next time. If, though the teaching has been good, and the pupils have worked diligently, the results are poor, then

it is probable that the material selected is too difficult for the class.

Thus, the teacher becomes critical of his own work and this is the way it should be. He becomes more critical of the course of study and is a more valuable asset to his associates in mathematics also in trying to make possible not only a better type of content material but also more efficient methods of teaching.

There is no good reason why a teacher who is qualified to teach mathematics at all cannot learn to formulate good tests to use in his own classes. I, for one, should favor raising the standard for teaching to insure this efficient both in content and in method.

Finally, this better use of tests has a stimulating effect upon the pupil. Under the old scheme tests have been looked upon by the pupils as something "cooked up" by the teacher to make them work or at best as a means of marking. Since in many schools great emphasis has been placed upon marks no end of dishonesty in taking tests has been resorted to. Often the pupil is conscious of no more wrong doing than is evidenced by many grownups who make a fine distinction between stealing and "swiping." The attitude of the pupil is that he must "get by" as he says.

I find that pupils are interested in this better use of tests. They enjoy taking them because it is clear that they are testing what they should have learned. The tests are a challenge to their spirit of competition and desire to excell. The chance to cheat on such a test is very small if the teacher is careful to explain the purpose of the tests and gives the pupil the opportunity to compete against his own former record as well as those of his classmates.

It is my hope that you may agree that such a use of tests is a better use and that it may help to usher in a new day in the teaching of mathematics in this country.

NEW TYPES OF SCHOOLS IN GERMANY AND THEIR CURRICULA IN MATHEMATICS[1]

By Dr. W. Lietzmann, Göttingen, Germany.

In many European countries, public instruction has either been changed or else completely transformed since the war. In some cases, as in France, this change has taken the form of a reversion to the humanities, the classic side. In others, as in Austria, the shift has been toward the modern scientific one. What is happening in Russia is beyond our knowledge. But even in the recently created border countries, people are ardently at work devising new types of schools. On every side, one hears of the reorganization of the curriculum.

The state of things in Germany has changed in noteworthy respects from the conditions of the time before the war as these were presented in the Report of the German Subcommittee of the International Commission on the Teaching of Mathematics,[2] and the statements which appear in the last Report of the National Committee on Mathematical Requirements[3] should be supplemented and changed in many respects.

The new constitution of the German Government provides for a uniform regulation of the school system for the entire country.[4] Thus far, however, only one important change has been made by law: the introduction of a four-year unified school (Grundschule)[5] through which all children must pass whether they are to continue in the Volksschule or whether they are to later go to the university. Formerly, attendance at the three-

[1] Submitted to *The Mathematics Teacher* by Professor David Eugene Smith; translated by Vera Sanford, The Lincoln School.

[2] *Abhandlungen über den mathematischen Unterricht in Deutschland*, edited by F. Klein, Leipzig, Teubner. 5 vols.

[3] J. C. Brown, *Mathematical Curricula in Foreign Countries, The Reorganization of Mathematics in Secondary Education*, The Mathematical Association of America, 1923, p. 129ff.

[4] Ibid. p. 139. ["In Germany centralization of educational authority is not carried to so high a degree as in France. Each of the German States is independent in local affairs, and the educational system is characterized by flexibility rather than rigidity."]

[5] [The term "unified school" though not a direct translation of the German name is frequently used for this type of school, and although it has not been standardized as yet, the name has been used in this translation as best conveying the meaning of a school through which all must pass whatever their academic destination."]

year primary school (Vorschule) was sufficient to obtain admission to the higher schools. It is understood in the Elementary Education Bill that the initiative in changing the school systems is left to the individual states for financial reasons. Nevertheless, one common line of development is compulsory.

The so-called unified higher schools (grundständigen höheren Schulen) continue to have their nine-year course, so that preparation for the university now covers thirteen years.[1] Hamburg is the single exception for it is satisfied with an eight-year secondary school. In addition to the Gymnasium, Realgymnasium, and Oberrealschule,[2] there has been added a fourth way to enter the university: the German Oberschule in which but two foreign languages are taught, and sometimes only one, while all the things that contribute to German civilization are emphasized. The Gymnasium, it will be remembered, is the classical school; the Realgymnasium, the modern language school; and the Oberrealschule, the mathematical and scientific one.

There is, however, an entirely different way to enter the universities and technical schools: one that closely resembles the system of the United States. After completing the unified school, the pupil remains in the Volksschule for three or four years and then he goes to a high school (Aufbauschule) which prepares him for the university in six years.[3] These schools are being founded in considerable numbers,—in Prussia, by the hundred. In their choice of studies they resemble either the Oberrealschule or more commonly the German Oberschule.

In almost all the German states, the normal schools (Seminare) whose function is the training of teachers for the Volksschule, either have been or are now being discontinued. In the future, all Volksschule teachers will go to the university but this improvement has not actually taken place to date in many localities, as for instance Prussia. Instead of the normal

[1] [Formerly, it was twelve. See the Brown Report.]

[2] There is no Bürgerschule running parallel to the Gymnasium, Realgymnasium, and Oberrealschule, as Brown asserts in the report cited above. There are, however, in Prussia for instance, the so-called Mittelschulen which give pupils from the Grundschule a limited five or six-year course. No decision for the future of these schools has yet been made.

[3] [This should be contrasted with the former order of things in Germany under which it was exceedingly difficult if not impossible for a pupil to arrange his course to enter the university unless that decision had been made before his fourth year in school.]

schools, several states including Thuringia and Saxony, contemplate having a course for teachers either in the universities or in institutes connected with them. But for financial reasons, the uniform regulation which the constitution anticipates cannot be put into execution at present.

Thuringia alone has completely broken with the past. Her system now consists of a unified school, a three-year elementary school (Unterschule), a three-year middle school (Mittelschule) and finally a three-year upper school (Oberschule). The elementary school offers two parallel courses differing in that the one course offers a modern foreign language and that the other is the old Volksschule which still retains its eighth year class for pupils who are going to go to work on graduation from this school. In the middle school, three parallel courses are provided, one of which has Latin as a second foreign language. There are four types of upper schools ranging from the Gymnasium, Realgymnasium, Oberrealschule to the German Oberschule. It cannot be denied that the execution of these school plans is contingent upon party politics. Consequently, one must be skeptical as to their permanence in the future, especially as opposition to these proposals is pretty common among the universities themselves. There are similar plans in Saxony, but these have not reached definite form as yet.

The secondary school course for girls, in Germany, has differed in many respects from that for boys. And though coeducation has been increasing in recent years, the separation of girls and boys is still general. As an example of the girls' schools, I will explain their usual organization in Prussia. Beginning with Easter 1924,[1] the four-year unified school will be followed only by the six-year Lyzeum which gives the pupils but a restricted course. There will be two ways to prepare for the university, however. On the one hand, a girl may leave the Lyzeum after three years and enter a six-year academy (Studienanstalten) which resembles the Realgymnasium, or on the other hand, she may finish the Lyzeum and then go to the Oberlyzeum and prepare for the university in three more years. These Oberlyzeen used to provide for the training of teachers of the higher schools for girls. This is over now, and a new regulation

[1] [The school year in Germany begins at Easter.]

of 1923 provides their free and unrestricted admission to the university.

These countless innovations and changes in the administration of secondary schools naturally necessitated new courses of study. These are to be expected in every line, but only the beginnings have been noted in the Aufbauschule, the Oberschule, the Lyzeum, and the Oberlyzeum in Prussia.

In mathematics these changes almost invariably conform to the suggestions made in 1922 in the proposed plan of the German Committee on Mathematics and Science Teaching.[1] I will list here only the salient features. We inserted a short intuitive course before the systematic geometry and we take care that the child shall become acquainted with solid geometry as soon as possible. Consequently we are using a moderate 'fusion'. The function concept will be stressed early in algebra and arithmetic and graphic representation finds extensive application. In the upper classes, the function idea occupies the foreground in the teaching of arithmetic. Aside from the theory of combinations which is seldom handled, the study of functions that are rational integral, rational fractional, algebraic, and transcendental, gives opportunity for the study of equations, maxima and minima, curve tracing, and infinite series. In connection with these, the differential quotient enters early in the last three years of the course in real problems which involve the application of mechanics i.e. velocities and accelerations. The general form of the differential quotient is not discussed, but simply x^n and perhaps $\sin x$ and $\cos x$.

The maximum course in Geometry is only offered in the Realanstalten[2] and there only with restrictions. It consists of a course in conics which allows us to contrast the different methods of geometry. First there is the stereometric introduction of the section of a circular cone, then the locus definition in a plane with the problem of Apollonius, then the union of both through the contribution of the Dandelin spheres. After this,

[1] *Neue Lehrpläne für den mathematischen und naturwissenschaftlichen Unterricht an den höheren Lehranstalten, nach den Meraner Lehrplänen von Jahre 1905. Schriften des Deutschen Ausschusses für den mathematischen und naturwissenschaftlichten Unterricht.* Second series, No. 8 Leipzig, Teubner, 1922.

[2] i. e., the Realgymnasium and Realschule.

comes an analytic study of conics with an investigation of the work of Desargues in which the section of a cone is treated as the projection of a circle. The theorems of Pascal and Brianchon, the study of poles and polars and collinearity come into the discussion which has as an optional last item the formation of conics through projective point systems and bundles of rays.[1]

For a long time, applied mathematics once even mathematical astronomy depending on spherical trigonometry, on the one hand, and projective geometry on the other, to name only the most important, have been greatly stressed in the university. These topics were inserted in the revised mathematics in the publications of the German Committee which designated the particular problems for each year, and the subject matter that remained was mentioned in connection with practise in surveying and mechanical drawing. The new plan of study is in harmony with these proposals.

People are endeavoring even more than before to bring the cultural meaning of mathematics into prominence for mathematics remains the beau ideal of a deductive science. Its logical and axiomatic aspect is made to appear more worth while in the later classes, naturally not in the beginning ones. It also offers an easy approach to philosophy e.g. to the study of logic. A consideration of the history of mathematics is particularly apt in the development of the cultural aim.[2] So perhaps more than ever before we are bringing the great problems into the foreground even though it be at the cost of the systematic completeness of certain topics.

Above all, we endeavor to use such problem material as will further the child's intellectual initiative. The dogmatic development method which is the rule in the universities has been abandoned in the secondary schools. A give and take of questions by the teacher and answers by the children gives the pupils an interest in the acquiring of new subject matter. Newer theories which are due to Gaudig, a Leipzig teacher who died recently, urge the teacher to efface himself to a considerable extent and to allow the questions and answers of the pupils to

[1] For a survey of the subject matter in mathematics treated at present in Germany, and for the methods employed, see,—W. Lietzmann, *Methodik des mathematischen Unterrichts.* Second edition, volume 2, Liepzig, Quelle und Meyer, 1923.

[2] W. Lietzmann, *Erkenntnislehre in mathematischen Unterricht.*

determine the sequence of topics. This is not always possible. Sometimes, however, this innovation shows good results.

This picture of post war times does not augur badly for the future of the German school system. The only question is how far can we go in the face of unfavorable economic conditions in carrying out these changes and in making what is now mere experiment take definite shape. The more so since the pedagogical magazines which offer an opportunity for the discussion of Europe's school situation must be slowly but surely sacrificed to social conditions. After all, mathematics has an advantage over her sister-subject natural science as the execution of her proposed method is not based to so great an extent on the purchase of numerous and costly apparatus. And finally, one does hope for better times!

A NUMBER PLAY IN THREE ACTS[1]

ACT ONE.

The curtain rises upon a home scene. Three children, a boy and two girls are working their problems at a table near the center of the stage. The mother is sewing near by and the grandfather is reading. The children are Jane, Evelyn, and Willis.

Jane—I can't get this problem.

Evelyn—Neither can I. The work is awfully hard these days.

Willis—Oh, girls don't know much about mathematics anyway.

Evelyn—If you're so smart let us see how much you have done.

Willis—Oh, I do all my problems in my head.

Jane—Work this example then.

Willis—(After a long struggle) You girls make me tired. Work your own problems.

Jane—He can't do it.

Evelyn—I knew he couldn't.

Mother—Here, here, children, stop your quarreling, and get to work; it will be bed time in ten minutes.

Evelyn—But mother, we will never get our work done in ten minutes. We have an awful lot to do.

Grandfather—Well, hurry and finish, your work and I will tell you a story.

Willis—Great.

Evelyn—Oh, that will be fine.

Jane—I'll hurry. (Children work steadily for a short time.)

Evelyn—I've finished.

Jane—Just a minute and I'll be through.

Willis—Oh, I'll stop.

Grandfather—Well are you ready for your story?

Children—Yes we're ready.

Grandfather—Well, while you have been fretting over your lessons, I've been thinking. About how long do you think it took the world to learn to count?

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[1] Dramatized by an eighth grade mathematics class in the Ben Blewett Junior High School, St. Louis, Mo. Based upon "Number Stories of Long ago," by David Eugene Smith. Contributed by T. Schlierholz.

Willis—Oh about ten years.

Evelyn—Oh more than that.

Jane—I guess it took a hundred years.

Grandfather—Well you are all wrong. It took a great many hundred years. It is so long ago that not even the wisest men of China can tell the year or even the century in which little Ching, the king's eldest son, played at the foot of Mount Yu, and told the soldiers who guarded him, what a lot of turtles he had. He really had only three, but he did not know a name for three, so he just said, "One, two, a lot."

Willis—Did they never need to count any more?

Grandfather—No. We use our numbers principally in trading. The people in those days did not need numbers further than that because they did not trade. The kings had many slaves and they sent them out to kill animals and bring them back for food and clothing.

At the time that Ching was growing up in China, there was another little boy named Anem, who lived in the plains of Mesopotamia, in southern Asia. He helped to tend his father's flocks, and drove away the wolves that prowled around at night. One day, Bel, his father, called out to Anem. "There are a great many sheep out there; drive them back." There were only a few, but Anem could not count past three, so he said, "one, two, three," and then "many."

There was also another boy who lived on the banks of the Nile, in a house grander than any he or his father had ever seen. It really had only one room.

Jane—Only one room?

Grandfather—Yes, because that was before people had real houses. He felt very proud when he said, "What a lot of palm trees we have around our house." There were really only six, but neither he nor his father could count past four, so he would just say, "One, two, three, four," and then "a great many."

Evelyn—Well did they never learn to count past four?

Mother—Oh yes, but it took them many years to learn to count further. The Chinese boys learned to count by twos. "One, two, two and one, two twos, two twos and one, two twos and two, a lot."

And the Babylonian boys counted by threes; "One, two, three, three and one, three and two, two threes, two threes and one,

two threes and two, three threes; three threes and one, three threes and two, many."

The Egyptian boys counted by fives: "One, two, three, four, five; five and one, five and two, five and three" and so on, to "five fives and four."

Willis—Oh, he was the best of all; he got to twenty-five.

Grandfather—Yes, it took the world a long time to learn to count. Some people learned to count by twelves, and we still have twelve inches in a foot, and twelve things in a dozen.

Willis—And some people say twelve ounces in a pound.

Grandfather—Yes these things have staid with us. In warm countries near the equator, where the people wore no shoes they counted their toes as well as their fingers. They had names for all the numbers through nineteen, and then they would say, "man finished."

Jane—We, still say score and that is twenty.

Evelyn—And the French say four twenties and that is eighty.

Grandfather—Yes that is true. But the reason the whole world today counts by tens is that the Chings, and the Anams, and you and I, and all the children of the world have ten fingers. The early Romans used their fingers in counting. (Shows them how.) I. II. III. IIII. V. VI. VII. VIII. VIIII X. Later they used letters to represent these numbers.

Evelyn—This is all so interesting, do tell us some more.

Grandfather—No, not tonight, but tomorrow night I will tell you another story. And now we must away to Dreamland.—Curtain.

ACT TWO.

The rear wall of the stage is covered with charts showing how numbers were first represented by various peoples. There are also charts showing how people multiplied and divided. These charts are carefully explained by the characters as necessary. Blackboard, chalk, and pointer are at hand. Enter—Grandfather, Mother, and children. Ancients are hidden behind curtain drapery.

Jane—Why what a strange place!

Willis—I never saw anything so queer.

Evelyn—But where are we?

Grandfather—It seems to me that we are in the mysterious Land of Numbers. (Curtain rolls back, showing two groups Egypt, Arabia, Hebrew. Multiplication, and Division on one

side; China, Greece, Rome, Fractions, and Decimals on the other side of the stage.)

Evelyn—Why this looks like an Egyptian. (Leads her to the center of the stage.)

Egypt—Yes I am Ancient Egypt. I was living four thousand years ago. Our people were the first to know much about mathematics. Our tax collectors had to know how to measure the land, because each time the Nile overflowed its banks, it changed the shape of each farmers land. We thought that we knew how to find the areas of the very shapes that bother you today, and strange to say our answers were nearly correct.

One of our greatest mathematicians was Ahmes. When he was a small boy he used to gaze at the Temple walls, and wish he could read what was written upon them. One day when Ahmes saw a priest reading the numbers, he asked him if he would teach him to read and write these numbers. The priest told him to get some papyrus and writing fluid.

Willis—What is papyrus?

Evelyn—Oh, I know what that is. It is a water plant that grows in Egypt. The Egyptians made a kind of paper out of the leaves by pressing them together.

Egypt—Fine. You know almost enough to have lived when Ahmes did. Ahmes learned to write his numbers this way. (Chart p. 16) This is the way he wrote 27,529 (Chart p. 20). On the temple walls were numbers like these giving the dates of the great wars and the kings' reigns.

When Ahmes grew to be a man, he became a priest and wrote the first book on mathematics. It is believed that he copied a book written still earlier. One of the examples in Ahmes' book was, "Heap its $\frac{1}{7}$, its whole, it makes 19." (Chart). You would say, "Add its $\frac{1}{7}$ to a number and the result will be 19." In algebra you write, $x+x/7=19$. Now when you make excavations in Egypt you find that mathematics was known by us much earlier than you had ever realized.

Jane—Why, this looks like a Chinaman.

China—Yes, I represent China. I am going to tell you how my forefathers wrote their numbers. It was many a long century ago, after the last of the Chings had learned to count by tens that there lived on the banks of the Yellow River a potter, whose plates and cups were known all over Shantung as the best to be

found anywhere in that great province. One day Chang watched his father write with a brush on a palm leaf, some marks that showed how many cups he had on the top shelf of his shop.

Now Chang had learned to count the cups, but he could write ten only by making ten marks. Chang said to his father, "I want to learn to write the numbers as you do." Then his father took the brush, moistened it, rubbed it on a small cake of black paint, and painted these strange figures. (Chart p. 14) This is the way he would have written 789 (Chart p. 14) This stands for 7, this for 100, this for 8, this for 10, making 280, and this for 9 making 789.

Now while Chang was learning to write, another boy named Wu, was learning to add and subtract with these numbers. He too found it necessary to use something like the Calculi, which the Romans used; but instead of pebbles he used bamboo rods. He thought that he was doing something wonderful if he added two numbers in two minutes, for you must remember that he had only a little pile of sticks to use. It took a long time to lay out one number and it took still longer to lay out two numbers, and then find their sum (Chart p. 52). This is the way 1267 would have looked laid out in bamboo rods.

It was many more than 1,000 years after Wu learned to use bamboo rods that we adopted the Roman idea of having the calculi fastened to an abacus, and so we invented our reckoning board, or "swan-pan." This we use in the schools, banks, and shops throughout all China even to this day, and we can do our mathematics more rapidly in this way, than some of you can do yours with pencil and paper.

Hebrew—I represent the Ancient Hebrews. We did not have much mathematics. Little Daniel lived at the foot of the Mount of Olives. He was a Hebrew. He had to count because he helped his father sell figs and other fruits at the market. The prices were written on boards or pieces of parchment and placed above the fruits. They did not need numbers above ten or fifteen. The figures he learned were the first few letters of the Hebrew alphabet (Chart p. 28) just as the first numbers that Hippias learned were the first few letters of the Greek alphabet.

Greece—I am Ancient Greece. Egypt may have been the first to measure the land, but I had my great mathematicians also. Hippias learned to write his numbers on parchment. This is the

way (Chart p. 26) he wrote 1, 5, 10, 100, 1000. If he wanted to write a number like 2,977 he had to use fifteen Greek letters. (Chart p. 26) Two thousand years ago another Hippias wrote on his parchment roll the Greek letters for numerals. (Chart p. 27) These little marks ' distinguish the letters from the numerals.

Thales lived 600 years before Christ. He introduced geometry into Greece. Thales was a merchant, and while he was travelling through Egypt, he became interested in mathematics. There were many queer stories told about Thales. One time when traveling through Egypt with a train of donkeys carrying salt, one of them slipped and fell into the stream. When he got up his load was lighter. The next time he came to a stream he did the same thing. To cure the donkey of this habit, Thales put a load of sponges upon the donkey's back. This time he again slipped into the water, but on getting up he found that his load was heavier. You may be sure the donkey did not try this trick again. Thales amazed the king of Egypt by measuring the height of the pyramids by measuring their shadows.

We all remember Pythagoras, because he showed us how to find the hypotenuse of a right triangle. He showed us that if we add the square of the base to the square of the altitude, we would have the square of the hypotenuse. Of course he did this by geometry. Pythagoras lived about 500 B. C. The Pythagareans had formed a secret group. If any member of the group discovered anything interesting he gave it to the credit of the whole group. You can learn something of group spirit from the Pythagoreans.

Euclid lived about 300 B. C. He wrote ten books called the Elements of Geometry. The geometry we study today is almost the same as Euclid gave it to us. In fact, today when an English boy wishes to say he is studying geometry, he says he is studying his Euclid. You must find out something about Archimedes also.

I want you all to know that Greece had a great woman mathematician also. Her name was Hypathia. She succeeded her father Theon, as a teacher. Well, Greece did a lot to help along the study of mathematics, but I must confess that we got our start from Egypt.

Rome—The Romans also did their share for mathematics. We did not make any great discoveries like the Egyptians and the Greeks did, but we knew how to use these discoveries and kept alive our interest in mathematics. Our warriors and merchants visited these countries and brought back this knowledge of mathematics. In time we became the greatest nation of the world and helped to spread this knowledge throughout Europe.

Titus who became one of our great men, was taught the numerals that were used in the shops. This is the old way of writing the numbers (Chart p. 28) and this is the later way. (Chart p. 29) If Titus had been asked to write a number larger than 1000, he would have been very much puzzled. This (Chart p. 29) is how he might have written 2,752,899. This is how he added 275 and 369. (Chart p. 47).

Titus had a friend named Caius, who attended a business school and learned to add and subtract, the only operations that were considered necessary. They did not know how to multiply or divide as you do today. Caius used to add and count by drawing lines on boards, letting them represent units, tens, and hundreds. (Chart p. 48) He would then place pebbles on the lines. Caius spoke in Latin, and his word for pebble was calculi, so instead of saying "Caius pebbled his answer," you would say "Caius calculated his answer."

Anyone who had any business to do would carry a small bag or box of calculi around with him. They used to carry boards with grooves cut into them, where they could place their pebbles. Thus we had our first adding machine.

Arabia—Don't you want to hear from Arabia?

All—Oh, yes, do tell us about your country.

Arabia—Gupta was one of our great men of the early days. The priests soon noticed that Gupta was brighter than the other boys of the village, so they taught him the numbers up to four, for that was as far as they themselves knew. (Chart p. 35). This is the way he wrote them. He learned to write with a sharp piece of wood on a palm leaf. They wrote their books in this way. (Shows a sample.)

When Gupta was a little older he learned to write another kind of numeral, and taught them to other people. (Charts p. 37) No one could write a number like 207 because the zero was not invented until long after Gupta died.

In France, nearly 1000 years ago, there lived a boy named Gebert. He was a good student, so the priests sent him to Spain with a nobleman whom they knew. Here he met some Arabs who knew the Hindu numerals. He learned these and later when he went to Rome he explained them to others.

Not much is known about the very early Arabian and Hindu mathematics. Some manuscripts were found written on birch bark. These show that they had found out that $\pi=3.1416$. One of our interesting men was Baskara. He wrote a book on mathematics, and called it Liliavati, after his daughter. This book was written in verse. Here is one of the problems;—"Lovely and dear Liliavali, whose eyes are like fauns, tell me, auspicious damsel, what are the numbers resulting from 135 multiplied by 12? Tell me what is the quotient of the same product divided by the same multiplier?

The word "algebra" came from the name of an Arabian book on mathematics. The Arabs studied and treasured the mathematics of Europe and Egypt. Later the people of Europe were so much interested in wars that learning was neglected. Many manuscripts were lost or destroyed. When the Moors entered Spain they established schools there. Many men from other countries studied in these schools. Thus the Arabs were able to give back to Europe much of her knowledge. These manuscripts were translated into Italian, and the Hindu-Arabic numerals were used in these translations. In this way they were introduced into Europe. We think we gave the world a most valuable gift when we gave her these numerals.

Evelyn—(Seeing Multiplication and Division) Why, who are these strang people?

Jane—(Discovering Fractions and Decimals.) And here are two others:

Multiplication—I am Multiplication. I had a very hard time to become known. People did not know who I was for a very long time. They could only add and subtract with Roman numerals. Some people used the calculus, but that was too difficult. I shall tell you about three boys who wanted to learn how to multiply. They were Leonardo, Cuthbert, and Johann.

Leonardo lived in Pisa. He learned arithmetic from a Moorish teacher who could multiply very rapidly. Later Leonardo wrote an arithmetic. This is how he multiplied. (Chart p. 66) First

the multiplicand, then the multiplier above, and the product above that. You see it is just your way upside down.

Cuthbert, an English boy, also learned the new way of multiplying, as it was then called. He also wrote a book, and this is the way he multiplied. (Chart p. 67) He did not know the short way of multiplying by zero.

In Germany it was the custom for boys who intended to go into business, to attend an Arithmetic School. Johann Widman went to such a school and learned to multiply as the other boys did. Later he wrote an arithmetic. He was the first man to put + and — signs in a book. These charts show how multiplication appeared in the early Italian books. (Chart 70 carefully described).

Willis—Oh, these were easy ways and lots of fun.

Division—A long time ago, about the time of Ahmes, the people could not and did not know how to divide. They had a hard time with multiplication, but a still harder time with me. It took many years to learn how to divide, since the people had to know how to multiply first. I want to tell you a little story about a boy who lived at the time of Christopher Columbus. This boy's name was Filippo. He went to a small Italian school. One day the teacher asked Filippo to write two numbers on a small board which he used for his arithmetic. He told him to use the Roman numerals, and then to divide the first number by the second. Of course Filippo could not do this. He said, "I can do it if you will let me use the new figures."

If you had been there you would have thought Filippo's way very queer, for as he finished with a figure he would cross it out. This was called the "scratch method." This is the way it looked, (Chart p. 75) and this is the way he did it. (Division works a problem by the "scratch method" on the blackboard.)

If you had lived when the Pilgrims came to America, you might have learned to divide by the scratch method, because many people used it them. A few years ago I found that they still used the scratch method in Morocco and thought it better than your way. What do you think about it?

Evelyn—Well I think that I like our way best.

Fractions—(Writes on the board when necessary.) Since you have not heard about me, I shall introduce myself as Fractions.

and shall tell you how these ancient men learned fractions. Many thousand years ago, when Ahmes learned from the priests in the temple, how to add and subtract, he felt that he had not much more to learn. As he grew older, however, he found that he needed to know something about me. Egypt at that time had no fractions with a numerator greater than one, with the single exception of $\frac{2}{3}$. When the priest and Ahmes spoke of $\frac{3}{4}$, they thought of $\frac{1}{2} + \frac{1}{4}$, and when they spoke of $\frac{7}{8}$, they thought of $\frac{1}{2} + \frac{1}{4} + \frac{1}{8}$. So it is no wonder that Ahmes had a great deal of trouble in learning fractions, and it is no wonder that he never learned to use them as you do today.

More than a thousand years after Ahmes studied in the temple on the banks of the Nile, there lived in Alexandria at the mouth of the Nile, a boy named Heron. He was interested in machinery, and in measuring heights and distances. He made friends with the scholars in Alexandria who studied the stars. When he made very careful measurements, he found that he needed another kind of fraction. These were fractions that had 60 or 60×60 or $60 \times 60 \times 60$ for their denominator. Just as you write .5 without a denominator, he could write $^{23}/_{60}$ without a denominator. He wrote $^{23}/_{60}$ this way—23', and $\frac{23}{60 \times 60}$ this way—23'', and so on. Today you still say 23 minutes and 23 seconds, showing that you still use the fractions that Heron used.

There were other ways of writing fractions. How strange it must have been when people wrote fractions this way: L/LIII and VI/VIIII.

Decimals—(Using the blackboard when necessary.) Do you know who I am? I am Decimals. I am first cousin to Miss Fraction, but I am much more important than she is. It was only about three hundred years ago that I came into this great world. Simon Stevin went to a school in Belgium. Like Ahmes and Heron he wrestled with fractions, but while these two boys had despaired of ever learning to conquer them, Simon made up his mind that he would be the victor, and he was. Simon Stevin saw that the common fractions were alright for cases like $\frac{5}{8}$, $\frac{7}{8}$, $\frac{5}{12}$, but they were not good for much when it came to very fine measurements.

The world was much interested in science, especially astronomy, and so needed better fractions. Simon Stevin wrote the first book on decimals. This is the way he wrote 12.754; 1 2◎ 7① 5② 4③ You see that he did not use the decimal point that you use.

Decimals, like fashions, change, and so in different countries people write them in different ways. In England they write $3 \cdot 4$, with the point halfway up; in France and Germany, 3,14, using the comma; and in your country 3.14. And strange to say, you all call it three and fourteen hundredths.

Evelyn—Why, I never knew the story of mathematics was so interesting.

Jane—Well, we have had a real mathematics lesson.

Willis—Yes, but this was interesting!

Grandfather—You will find everything interesting, if you just look at it properly. And now we must say good bye to these interesting people and their wonderful land.—Curtain.

THIRD ACT.

Children working at table, left side of the stage. Grandfather and mother seated near center. All work done on board and explained when necessary.

Grandfather—So this is to be our last night.

Evelyn—Last but one.

Grandfather—Are you sure?

Children—Sure!

Grandfather—What shall it be tonight, square root or cube root?

Willis—Square root doesn't sound interesting.

Grandfather—Logarithms?

Jane—Never heard of them. Tell us something interesting.

Grandfather—Well, as we have a good log fire here, let us imagine it a cold night, and this place at the edge of a dense forest; the house being built of logs, and having a huge fire-place, with many comfortable chairs arranged around it. Over the heavy door, someone has written in the years gone by, these words, "Let no one without imagination enter here." So the door is closed to you and me and to all the world if we have not imagination. For us the log fire is cold and the candles do not burn, if we belong to that humdrum class that never has any day

dreams. You then without any imagination, who belong to those unfortunates, who believe only what they hear, stop on the threshold. For you the log fire has no attraction, and the empty chairs no occupants. (As Grandfather calls up these guests, they appear silently from each side, carrying stools and arrange themselves in a semi-circle facing the audience. The children, surprised, seat themselves on the floor, in the center of the stage, facing the strangers.) From out of the great unknown of time and space, they come, one by one to the house of logs, our friends of long ago. Chang from the Land of the Yellow Dragon; Anam from the Tigris; Menes and Ahmes from the banks of the Nile; Hippias from Greece; Titus from the Seven Hills of Rome; Daniel from the Mount of Olives; Gupta and Gerbert, and most of the others of whom we have read. Strangers to each other, strangers to the present civilization, but drawn together by their interest in the puzzles of the world of number. Now if you will watch and listen, perhaps our friends will tell you something interesting.

Chang—In my land there is an ancient book, perhaps the oldest in the world. The name of this book is "Yii King". It was written thousands of years ago, and records one of the earliest of all curious things connected with number. It says that once upon a time, there came out of the Yellow River, a large turtle, and on its back were strange marks, which puzzled everybody who saw them. These I have copied on paper for you to see. (Shows chart p. 97).

Gupta—Oh, I know what that is. It is a magic square and the dots are numbers.

4	9	2
3	5	7
8	1	6

The columns add to fifteen and so do the rows. Now watch the diagonals. It is the world's greatest number mystery, and is used as a charm all through the East and in the Middle ages, long after I died, it was used in many parts of Europe to drive away disease and bring good fortune.

Grandfather—Yes, Gupta is right, for in the Magic Square we have one of the oldest and most interesting curiosities to be found anywhere in the world.

Menes—When I was living in Egypt in the early days, we did not have many number puzzles. But since then I have watched

the world and have seen many strange things happen. One of the most curious relates to the numbers now used in Europe and America and all places under their influence. If you take any number, say 3467 and reverse it 7643 the difference will always be divisible by 9. Try this and see if it is not true.

Chang—I have it.

$$\begin{array}{r} 3827 \\ 2783 \\ \hline 9)1044 \\ 116 \end{array}$$

Daniel—You did that wrong. It should be 3872.

Menes—That does not make any difference. You may mix the figures any way you wish, and the rule still holds. After you have studied a little algebra, you will see how this is done, for algebra is like a great electric light; it reveals all the secrets of arithmetic.

Heron—One of the most curious things that I have noticed in looking at the world for the past two thousand years, is seen in a simple problem in addition. If you write any three numbers, say numbers of four figures each, I will at once write three numbers underneath, and before doing so will tell you the sum of all six numbers. It does not make any difference what numbers you take, nor does it make any difference whether they are all alike or all different.

Adrien—I will write these numbers.

$$\begin{array}{r} 7632 \\ 5847 \\ 6195 \end{array}\Big\} \text{ Written by Adrien.}$$

$$\begin{array}{r} 2367 \\ 4152 \\ 3804 \end{array}\Big\} \text{ Written by Heron.}$$

$$\overline{29997}$$

Heron—The sum of all six numbers will be 29997.

Evelyn—I am going to try that and see if it is right. (Adds the columns aloud.)

Heron—It would be fun for all of you to try it and see how it is done.

Gerbert—A strange thing was shown to me when I went as a boy to Barcelona. Any one of you may write any number of as many places as you please, and I will at once write a figure at the end of your number and make it exactly divisible by 11.

Jakob—I will write that number, (He wrote 74289).

Gerbert—(Writes 6 at the end of the number, and divides it by 11.) Now would you like me to show you how I did it?

All—Yes. Show us.

Gerbert—9 and 2 is 11, and 7 is 18. 8 and 4 is 12, and you must add 6 to make it 18. I have now given you good measure, for I have made this number exactly divisible by 22, 33, and 99.

Michael Stiffel—When I was a boy we had a couplet that ran like this, "Ten fingers have I on each hand, Five and twenty on hands and feet." You know we used to say five and twenty, for twenty-five.

Chang—But you have not five and twenty fingers on your hands and feet.

Cuthbert—You must count your toes as fingers.

Hippias—Even then we have only twenty.

Michael Stiffel—It is perfectly easy if you only know how to say it. Shall I tell you?

All—Yes, do tell us.

Michael—Ten fingers have I; on each hand five; and twenty on hands and feet.

Evelyn—Well that was a good one on us.

Jakob—I believe that one of the oddest things that I ever saw in numbers, is a set of products which I shall show you.

$$3 \times 37 = 111$$
$$6 \times 37 = 222$$
$$9 \times 37 = 333$$
$$12 \times 37 = 444$$

Now can anyone tell me what 15×37 is?

Willis—I guess it will be 555.

Jakob—You are right. Another strange thing is that $1 + 1 + 1$ is 3; $2 + 2 + 2$ is 6; $3 + 3 + 3$ is 9, and so on.

Anam—While you are talking about such curious things, I will show you something really interesting.

$$1 \times 8 + 1 = 9$$
$$12 \times 8 + 2 = 98$$
$$123 \times 8 + 3 = 987$$
$$1234 \times 8 + 4 = 9876$$
$$12345 \times 8 + 5 = 98765$$

and so on. This is how it looks when it is finished. (Shows chart p. 106). Now when we go home, try it and see if it is correct.

Menes—Yes that is very interesting, but I will show you something quite as curious.

$$1 \times 9 + 2 = 11$$
$$12 \times 9 + 3 = 111$$
$$123 \times 9 + 4 = 1111$$
$$1234 \times 9 + 5 = 11111$$

If you continue this until you have used all the numerals, your products will still contain only ones.

Ahmes—It seems that these figures which were invented many hundreds of years after I died are very queer indeed. You could never have done that with the figures that I learned four thousand years ago.

Cuthbert—Yes, and every one knows that no one else could do anything with them, they were so bad.

Ahmes—Well you need not be so proud of your numerals. Try writing twelve thousand, twelve hundred, twelve.

Cuthbert—Why, that's easy (Writes 121212.)

Ahmes—Now see what you have written.

Cuthbert—Oh that's 121,212. Well I guess it can't be done.

Ahmes—In the earlier days we would have written it something like this: 12M 12C 12. Of course we would have used other marks for the numbers, and signs for thousand and hundred.

Titus—When I was a boy, I was given a puzzle problem, which I will have to change a little so as to give it to you in English: 6—9 9—10 40—50 Can any of you do this so as to have six for a remainder?

Children—It can't be done.

Titus—Yes it can.

SIX	IX	XL	
IX	X	L	
S	I	X	spells six.

Adrien—Here is a good one. Show me how to write one hundred using the nine digits and the signs of arithmetic.

Hippias—Impossible.

Adrien—I will show you. $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 \times 9$ equals 100.

Daniel—It does not equal 100. (Adds all the numbers through 8 and then multiplies the result by 9 and gets 324.) And 324 is not 100.

Adrien—You did that wrong. 1, 3, 6, 10, 15, 21, 28; $8 \times 9 = 72$; 72 plus 28 is 100. You do not know that you must multiply first.

Daniel—Oh, I forgot.

Gupta—Who can write four with four nines.

Hippias—That is too easy. Why don't you ask who can write twenty using four nines. That is a problem worth thinking about. I will show you all of them.

$$\frac{9}{9} + \frac{9}{9} = 2 \qquad \frac{99}{9} + 9 = 20$$

Can anyone write 100 without using any zero's?

Cuthbert—$99 + \frac{9}{9} = 100$. Now I have shown you how to write an even number using only odd digits.

Ahmes—Now before we go, I am going to tell you a good one. Two fathers and two sons divided three apples equally amongst themselves, so that each received a whole apple. How was this possible?

Hippias—They had to cut the apples.

Ahmes—No they did not. Shall I tell you?

All—Yes tell us.

Ahmes—One was the grandfather, one the father, and the other the son. Were there not two fathers, and two sons? (The clock strikes twelve and the visitors disappear. The children return to their work and look up startled to find themselves alone.)

Evelyn—Why, where are all our friends?

Grandfather—They have gone. They came because you had good imaginations. Now I only hope you will remember all the interesting things they have shown you. The evening is late and we must say good night.

END.

PURPOSE, METHOD, AND MODE OF DEMONSTRATIVE GEOMETRY

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Preparation for living is the conception of the purpose of education which is most popular,—with emphasis often if not usually upon a narrowly specific preparation for some vocation. "Practical value" (usually undefined) has come to be a shibboleth used for a subject, and any apparent lack of it is used against a subject. Teachers are asked to indicate what parts of their subjects, if any, contribute certain specific abilities or qualities which are needed by adults; the implication is that the content is to be determined by what adults actually will use within or outside their callings. In this manner, emphasis is thrown upon specific preparation for the future in terms of subject matter.

While no one will deny the importance of teaching children any facts, or of developing in them any skills which they are certain to need in adult life, it does seem futile to attempt to build a curriculum in any such manner. First of all, the specific future needs of children cannot be known; secondly, when known, they vary so much that only individual instruction or instruction to small groups would be inevitable; thirdly, for the mass of workers, few and most limited skills are necessary in the modern factory where the "iron man" (See Arthur Pound's *Iron Man in Industry*)—the automatic machine-supplies almost all the intelligence needed.

Tested by this criterion of *specific preparation in facts or even in skills,* it is impossible to defend much of the instruction in the academic subjects or even in the so-called vocational subjects. In this respect, secondary mathematics as a whole, and demonstrative geometry in particular, is no exception, since the majority of people do not need much geometry either within or outside their callings. Some few geometrical truths are needed by all people; some skill in constructing geometrical figures is needed by a still smaller number of people; ability to make indirect measurements either by scale drawings or by some kindergartenized form of trigonometry is needed by an equally small

[1]Presented before the Mathematics Section of the High School Conference at Urbana, Illinois, November 23, 1923.

number of people. But all this can and possibly should be taught inductively rather than deductively, and is no argument for *demonstrative geometry*. The sooner this fact is admitted the better. Possibly this explains the surprising and regrettable action of the National Committee in placing demonstrative geometry in the list of elective courses, except for a small amount of it which they recommend as part of the work in grades 7, 8 and 9. Certainly this explains the action of many schools in making demonstrative geometry elective.

If this narrow conception of preparation for life is to be applied to mathematics, it should be applied equally to all subjects. As a matter of fact, a broader conception is applied to many subjects, and by some people to mathematics. Besides leading pupils through the little mathematics they *will need* as adults, and the little more they *may need* as preparation for further schooling or for some vocation, *an effort should be made to discover for them any capacity for and interest in mathematics which they may possess, since one of the major purposes of the secondary school is to discover the dominant interests and powers of the pupils. Demonstrative geometry is most peculiarly fitted to do this for mathematics,* as it is, of all the elementary courses in mathematics, the one in which pupils receive training in the "science of necessary conclusions," which has been offered as a definition of mathematics. Algebra even when enriched by emphasis upon the graph, the equation, the formula, and generalities about the function concept does not give this training; neither does intuitive geometry or numerical trigonometry—as valuable as all these may be as educational media.

Next, while a large appreciation of the place of geometry in the work of the world, especially in the field of design, can be secured independent of demonstrative geometry, an *intelligent appreciation* of the place of geometry in constructing geometrical designs (*e. g.,* Gothic windows) is independent upon understanding that these constructions are proved correct by processes of demonstrative geometry.

So much for the knowledge benefits contributed by demonstrative geometry. There remains its contribution to that other large responsibility of the school, namely, developing "desirable habits of thought, achievement and conduct." (Hanus.) The pendulum of discussion of formal discipline or transfer of train-

ing has swung again. Psychologists almost unanimously agree that some transfer of training takes place. Read the section by Miss Blair in the report of the National Committee; or, the monograph by John E. Coover on "Formal Discipline from the Standpoint of Experimental Psychology." The latter, in his summary, states that objective evidence, and also introspective, discloses transfer of large amount, *especially of habits and ideals*, and that it is quite probable that there is even greater transfer, especially of children of secondary school age, who study for a long period of time, under proper schoolroom conditions, as complex a study as demonstrative geometry. Recently, Professor Thorndike made a preliminary and tentative report on an investigation designed to determine whether the study of the customary school subjects does influence general intelligence, and to determine which studies, if any, exert the greater influence. He reported that there *is* evidence of increase in general intelligence; that this increase *does* seem to vary with the subjects studied; that algebra and geometry apparently belong in the second group, when subjects are ranked in the order of their influence on general intelligence—being preceded by general science, and being followed more or less remotely by English, history, and all of the so-called vocational courses.

Fix in mind the phrase "desirable habits of thought, achievement and conduct." Observe the word *habits*. The secret of effective education is the formation of habits. The content as well as the methodology of arithmetic is being stated in terms of habits to be developed. In the secondary school also we must develop habits, for all of us are creatures of habit.

Observe the kinds of habits,—habits of thought, achievement, and conduct. *Demonstrative geometry uniquely develops the habit of deductive thinking*,—more important than the "habit of functional thinking." To be sure this habit is based upon appreciation and use of it in geometry, but there is reason to believe that the ideal and habit are transferred elsewhere. No other subject in the curriculum, and no other part of elementary mathematics can fill the place of demonstrative geometry in developing this habit and ideal. *Unless this function of demonstrative geometry is recognized as paramount*, then *demonstrative* geometry loses its largest claim to usefulness in a scheme of general education.

Next, there are habits of achievement, habits of doing things. These to be sure are indirect benefits of the study of any subject; they are especially numerous from the study of geometry. Among them are habits of learning. Wisely, indeed, has some one said recently that the chief gain to be acquired in school is not learning something, but *learning how to learn.* The whole tendency (as said previously) seems to be in the direction of learning *something,* with emphasis on the *something.* But that is unimportant to one who has *learned how to learn.* This last acquisition enables the fortunate possessor to learn facts readily or even to acquire skill wherever they are needed,—if given time. Not the least subdivision of this is learning how to read a book. Demonstrative geometry affords *unequalled* training in this quality. In most subjects, a pupil reads to *learn something;* in a geometry, a pupil must *read and reflect.* His judgment is challenged at every step by the necessity of supplying the authority for every statement made. Concentration on and filling in the argument is a prime necessity. *No other text that he studies* gives him this training in kind, let alone in degree. In the solution of the originals of demonstrative geometry, there is unexcelled training in concentrating upon, in attacking, and in solving a baffling intellectual problem *where exactness of argument* is demanded. Demonstrative geometry not only furnishes an unexcelled list of graded problems but also opportunity for exceptional training in methods of solving them, so that pupils can have the joy and the benefit of *"successful experience."* There are habits of organization and presentation of material. The training of demonstrative geometry is not unique in this respect, but it is unexcelled. The demand for exact definition and use of words, for careful reading of sentences, and for adequate but concise expression of an argument gives exceptional training in English expression. The presentation of this argument before the class gives training in public speaking. The presentation of this argument in writing gives training in organization of material. The drawing and construction of figures gives "manual training" which all pupils need, and of a form they do not get elsewhere.

Finally, there are desirable habits of conduct to be developed. In what other study is regularity and completeness of meeting daily responsibility more necessary? In class, the highest forms

of good social conduct are demanded; attention and open mindedness in listening to another's argument; helpfulness in aiding another to understand an argument; cooperation in filling in weak spots in an argument.

As real as all these values may be, when it comes to a question of the proper mode of approach, none of them offers the strongest appeal to the learner. Over twenty years ago, an instructor of mine said that *education is not preparation for life, it is life.* Our pupils are not interested in preparing to live; they want to live happily now. Curiously, a foreign critic agrees with them when he says that we concern ourselves too much with preparation for the future. As a matter of fact, right living now, intellectually as well as physically, is the best preparation for living in the future. Let us try to teach demonstrative geometry so the pupils will get intellectual pleasure out of it; try to have them study it for the fun there is in it; try to lead them to this great science as to an adventure which they are first of all to enjoy; try to stir up their curiosity about and interest in this wonderful product of the human intelligence, "the finest example of the highest type of thinking of which man is capable", the subject which has been taught longer and in more schools than any other subject of the curriculum; try to stir up among them a desire to try out their intellectual muscles on truths,—interesting truths which challenged some early discoverer and have since challenged but been mastered by thousands of learners; try to lead them on to intellectual combat and conquest, two appeals which are far stronger in youth than any promise of future usefulness. *Play up the pleasures rather than the profits of the study of geometry,—and then make good on it.* Be an enthusiast for geometry, and then help the pupils master a rich course in geometry. This ideal will carry our classes along much farther than that of preparing them for the future. Is it visionary? Not at all. The battle is indeed half won because, in spite of all talk to the contrary, independent investigations have disclosed that boys and girls do like their geometry. They like it a little less well than they do the algebra; the boys like it a little more than the girls; both like it about as well or better than their English; both like it better than they do their history. The reason given by the majority of them for liking it is not that it has practical value but rather that it is interest-

ing. If we read between the lines or gather additional evidence from them, we learn that this means that the teacher is enthusiastic, skillful, has an agreeable personality, and that the subject is one in which the pupils can succeed when they try.

As to methods of instruction and modes of class management, —the foremost need is a course in Intuitive Geometry, preceding the Demonstrative Geometry, to provide a foundation of concepts, nomenclature, and notation; to provide training for the hand in the use of geometrical tools; to provide training for the eye in seeing geometrical relations; to provide training for the mind in making generalizations from observed or measured relations. The subject matter of this course can be limited to the most commonly used geometrical facts, especially to the mensuration theorems for the most common plane and solid figures, and to the study of the fundamental concepts of geometry involved in these figures. The danger at present is that the life of the course in demonstrative geometry will be sapped by teaching intuitively too many of the truths formerly taught in demonstrative geometry. Thus, it seems most questionable to me to teach anything about congruence, or similarity or even of constructions with compasses in the intuitive geometry. The course in intuitive geometry should come in the junior high school, and seems to be establishing itself there. If there is not such a course preceding the course in demonstrative geometry, the first two to four weeks of the latter course should be chiefly intuitive geometry with gradual increase of the deductive element.

Next, the only way in which to get the pleasures and profits out of demonstrative geometry is to teach demonstrative geometry,—not architecture, or designing, or surveying or anything else. It cannot be made incidental to something else. It must be taught deductively. This calls for maturity. There is much truth in the warning of Professor Sachs, I believe that it is dangerous to shove senior high school subjects down into years of further immaturity. Children in grades 7, 8, and 9, can do some kind of deductive geometry to be sure, if they are led along nicely by the hand, but they lack the powers of concentration and persistency demanded for the kind of demonstrative geometry that will give the benefits claimed in the earlier sections of this paper. Moreover, the peculiar character of this demonstrative geometry demands a considerable period of intensive study

of it if pupils are to get these benefits. Therefore, a continuance of the practice of concentrating on demonstrative geometry in the tenth grade is most desirable. The argument that younger children study the subject in foreign schools is not as weighty as it might seem. In the first place, conditions here are so different from those in any other country; in particular, nowhere else in the world is the populace taking advantage of secondary education as they are free to do and are doing here. But more important there is the question of ideals. Two teachers may claim that they are doing the same kind of work when in fact they are doing quite different kinds of work. If the detailed lesson on the Pythagorean theorem given in the report by Lietzman (Report of the German Sub-Commission, of the International Commission on the Teaching of Mathematics) as a sample of the mode of instruction in a German Oberrealschule is indeed a good sample, then it is not the kind of instruction in demonstrative geometry that I am talking about; it is indeed a good example of the kind of instruction that becomes necessary when the subject is taught to children of grades 8 or 9. However, any one who knows anything at all about conditions in our average public high school will not recommend instruction in demonstrative geometry earlier than grade 10,—especially for brief periods of time.

A text should be used in order that pupils may learn to read a book and in order that time may be saved for doing the many things that can and should be done in the course. This text should be one with incomplete demonstration, in order that mere memorization is impossible. It will furnish the standard for all the pupils, and the minimum of training for everyone in the class. To obtain the benefits of use of the text, the pupils must be taught that they can and must master the demonstrations in it. First, of course, they must be led to wish to do so, but as an alternative they must be compelled to do so. They can do much better in this respect than they usually do if we stop babying them. Be pleasant about it but be firm. As a matter of fact, pupil's do not dislike study under such instruction, much as they may squirm about at times, for they realize that they are getting results, and that is one of the potent influences producing interest. The training in independent demonstration should come from the solution of originals, and this must be made

the chief aim of the course. You may think that this is already being done, but there is reason for believing that it is not. Seek devices which help the pupils to solve the originals. Try to stir up their pride in doing what many tell them is difficult. The particular devices will vary, of course, with the teacher and the text. A minimum seems to include instrutcion in: *a* planning a demonstration; *b* fundamental principles of demonstration; *c* various methods of analysis, such as analysis of construction problems, analysis used in problems involving loci, algebraic analysis (one of the finest opportunities for correlation), analysis of demonstrations which involve numerical relations among the sides of the triangle; *d* the different types of proof, such as the superposition proof, the direct, the indirect, and the analytic. Try to habituate each of these devices by drill upon them in the solution of exercises. Time does not permit illustration of these various devices.

In order to develop desirable habits of expression, insist upon having oral and written arguments in full, with complete statement of authorities. Have the demonstration arranged in numbered steps under classifying heads. Have the figures drawn carefully. Have the pupils go to the board when giving a demonstration, in order to get experience in public speaking. Have a combination of class, group, and individual instruction. Have class instruction on fundamentals and for summing up reviews; have group work, possibly, for routine drills and preliminary reviews; above all have individual work to provide for development of individual capacity. Whether or not to have classes subdivided on the basis of intelligence testing is an open question. In the average small school it is futile to talk about the matter because there are not enough pupils to make more than one class. It seems unnecessary, if not unwise, even in larger schools, because a skillful teacher can easily outline special assignments for bright pupils in a class and in that manner adequately provide for the individual differences which exist. It is especially important just now to caution against doing only the minimum course outlined by the College Entrance Board. That might well be regarded the minimum for passing the course but it should never be considered as offering adequate instruction for even the average or upper third of the class.

THE TEACHING OF LOGARITHMS

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Since the subject of logarithms is recommended by the National Committee to be taught in High Schools, many teachers will be required to teach it who have not had a great deal of experience either in the use or teaching of that very interesting and time saving instrument of mathematics.

It shall be the purpose of this paper to present a few ideas, some old and some new and original, which have been found helpful in teaching logarithms. These suggestions differ from others chiefly in the more carefully graduated presentation of the material—interpolation being postponed until the student has acquired some ability to use logarithms in simple computations—and in the more simple determination of the characteristic.

In order to master logarithms the pupil must know (1) how to find the logarithm of a number, (2) how to use that logarithm to attain certain desired results, and (3) how to find the anti-logarithm. In order to have the student master these processes the teacher is confronted with three problems: (1) to develop an appreciation of the meaning of the logarithm as an exponent of ten (or e) and thereby a knowledge of why certain manipulations yield certain results, (2) to teach the student how to determine the mantissa and the characteristics of a logarithm, (3) to teach interpolation. Interpolation is a part of the process of finding the logarithm, but is not easy for the student to grasp along with the many other new ideas. Therefore by leaving it until last the student will have fewer new things to become familiar with at one time.

One reason for the great difficulty which students have in digesting and assimilating logarithms is the great number of new terms and processes which are usually mixed up in the first prescription. Such terms as logarithm, characteristic, mantissa, anti-logarithm, interpolation, cologarithm, incommensurable exponents, natural system, Naperian system, exponential equations, when all introduced in one lesson are apt to give even the brightest student educational indigestion. These nutritious morsels must be carefully presented, not too many at

once, so that the system of the patient may become adjusted to its new food and benefit thereby.

To master all these new terms and to use them intelligently is a stupendous and often fatal task. It is only common sense to say that if many of these terms can be omitted from the beginning of the presentation, the processes will be more easily mastered. In the plan here presented care is taken to present only as many new terms and ideas as efficiency demands, to drill on these for mastery before other ideas are presented, and to have the one rule so simply worded and so rationally developed as to make it most easy to remember.

I. *Logarithms as Exponents of Ten.* The explanation of a logarithm as an exponent of ten is so well done in many textbooks and logarithmic tables, that it will be unnecessary to discuss this extensively here. It is necessary, however, to explain this thoroughly to the students and to see to it that it is thoroughly mastered. Otherwise your students will be mere manipulating machines, they will not understand what they are doing, and will be getting merely the husks of education. The study of logarithms can be very well motivated by its introduction thru the applied theory of exponents. For instance, $a^2 \times a^3 = a^5$, and $(a^2)^{20} = a^{40}$ suggest short cuts in computation that can be readily developed.

The logarithm of a number is the exponent of ten required to produce that number. That exponent may be positive or negative, integral or fractional. Logarithms will usually be in this form, 2.75468. If $10^2 = 100$, then the logarithm of 100 is 2. If $10^{1.5} = 31.6$, then the logarithm of 31.6 is 1.5. If the logarithm of 17.8 is 1.25, that is merely another way of saying that $10^{1.25} = 17.8$.

In dealing with exponents we learned that in multiplying we added exponents, *e. g.*, $a^2 \times a^3 = a^5$. Therefore in multiplying 31.6 by 17.8, since these numbers are equal respectively to $10^{1.5}$ and $10^{1.25}$, all that we need to do is to add the logarithms and then find the number whose logarithm is this sum. For convenience the abbreviation "log" is used for logarithm.

log 31.6 = 1.5		log 43.6 = 1.63949	
log 17.8 = 1.25		log 5.58 = 0.74663	
the sum	2.75 = log 562.4	sum	2.38612 = log 243.288

After a few problems like this have been worked the student will naturally ask, "How do you know that the log 5.58 is 0.74663?" The logarithms of all numbers are given in a little book called a table of logarithms. In order to make the book as small as possible, however, a plan has been used which we shall have to study carefully before we can find logarithms from the table.

II. Finding the Logarithm—Characteristic and Mantissa.

In order to understand fully the logarithm we must know the effect of the fractional part and also of the integral part. Let us analyze a few simple logarithms.

$$\sqrt{10} = 10^{\frac{1}{2}} = 10^{.5} = 3.1623$$
$$10 \times 10^{\frac{1}{2}} = 10^{1.5} = 31.623$$
$$100 \times 10^{\frac{1}{2}} = 10^{2} \times 10^{.5} = 10^{2.5} = 316.23$$
$$1000 \times 10^{\frac{1}{2}} = 10^{3} \times 10^{.5} = 10^{3.5} = 3162.3$$

From these facts it is evident that

$$\log\ 3.1623 = 0.5$$
$$\log\ 31.623 = 1.5$$
$$\log\ 316.23 = 2.5$$
$$\log\ 3162.3 = 3.5$$

The logarithms differ only in the integral part and the numbers only in the position of the decimal point. Therefore the fractional or *decimal part of the logarithm determines the integers* that make up the number and *the integral part determines the position of the decimal point.* This is reasonable since $316.23 = 100 \times 3.1623 = 10^{2} \times 10^{.5} = 10^{2.5}$. Thus adding 2, to the logarithm coresponds to multiplying by 100 or moving the decimal point 2 places to the right.

We have certain technical names for the two parts of the logarithm. The integral part is called the characteristic, and the decimal part, the mantissa. The characteristics in the four logarithms given above are 0, 1, 2, 3, and the mantissa in each is the same, .5. Log 43.6 = .639, 1 is the characteristic and .639 is the mantissa.

The logarithmic table is really not a table of logarithms at all, but a table of mantissas. In using the table then you can find only the mantissa and you must determine the characteristic by inspection. Turn to your tables. (Five place logarithm tables.)

The first column gives the numbers with three digits, and the second column the corresponding mantissas. Find 187 in

the first column, see that its mantissa is 27184. The mantissa of 635 is 80277, of 977 is 98989.

The other columns give the mantissas where the fourth digit is given. Thus the mantissa for 1871 is 27207, for 1872 is 27231, etc., for 6358 is 80332, for 9772 is 98998 and for 9776 is 99016.

It is a simple matter to find the mantissa then for any number with four or less digits in it. It requires merely looking in the proper column and row. The determination of the characteristic is still more simple. So many presentations have two or three rather complicated rules which are not easy to remember. The one rule given here is very simple and easy to remember.

When the characteristic is zero—as in log 3.1623 equals 0.5—then the decimal point comes after the first digit. This is reasonable since we know that $10^{0.5}$ is between 10^0 and 10^1 or between 1 and 10. Moving the decimal point two places to the right would be multiplying by 100 or 10^2 and therefore 2 would be added to the logarithm. Thus log 316.23 = 2.5 since moving the decimal point two places to the right is the same as multiplying by 100 or 10^2 and since 316.23 is between 100 or 1000 it is reasonable to expect its logarithm to be 2 and a decimal fraction.

$$\begin{aligned}
\log\ 1.87 &= 0.27184\\
\log\ 187. &= 2.27184\\
\log\ 6.358 &= 0.80332\\
\log\ 63.58 &= 1.80332\\
\log\ 635.8 &= 2.80332\\
\log\ 6358. &= 3.80332\\
\log\ 63580. &= 4.80332
\end{aligned}$$

From this let us formulate the one rule. The rule, however, is dependent on a definition which must be given first. The *natural position* of the decimal point is *after the first digit.* If our logarithm table is a real logarithm table and not a table of mantissas then the characteristic in each case must be zero and the decimal point must be in its *natural position, after the first digit.*

Rule: *The characteristic is the same as the number of places the decimal point is moved from its natural position.*

This rule holds for all numbers, those less than one as well as those greater than one, with the provision that if the decimal

point is to the left of its natural position then the characteristic is negative.

$$\begin{aligned}
&\log 9.776 = 0.99016\\
&\log 97.76 = 1.99016\\
&\log .9776 = -1+.99016 = 9.99016 - 10\\
&\log .09776 = -2.+99016 = 8.99016 - 10\\
&\log 977.6 = 2.99016\\
&\log 97760 = 4.99016\\
&\log 9{,}776{,}000 = 6.99016
\end{aligned}$$

The beginning student should not, however, be confused by the introduction of numbers less than one. Let him first thoroughly master what has been presented. Have him find the logarithms of several numbers, none with more than four digits however.

He should now be able to work many simple problems. Care must be taken in assigning problems so that there will be no numbers less than one, and no processes involving interpolation at this stage. One lesson assignment at least can be devoted to drill on the ideas already presented. Simple products of two digit numbers resulting in not more than four digits, unless the others are zero, should be given.

Type of problems for first lessons:

$$\begin{aligned}
47 \times 83 &= & 245 \times 86 &=\\
64 \times 98 &= & 95 \times 874 &=\\
76 \times 29 &= & 765 \times 480 &=
\end{aligned}$$

Students should be taught to keep their work in good form to avoid mistakes and facilitate checking.

$$\begin{aligned}
(1)\ &\log 47 = 1.67210 & (3)\ &\log 2.45 = 0.38916\\
&\log 83 = 1.91908 & &\log 86 = 1.93450\\
&\log 3901 = 3.59118 & &\log 210.7 = 2.32366\\
(2)\ &\log 64 = 1.80618 & (4)\ &\log 765 = 2.88366\\
&\log 98 = 1.99123 & &\log 480 = 2.68124\\
&\log 6272 = 3.79741 & &\log 367200 = 5.56490
\end{aligned}$$

After finding the logarithms of numbers to be multiplied add the logarithms, and then find the number having the mantissa indicated. Thus in (1) the mantissa is .59118, and the table shows the corresponding number to be 3901. The characteristic

being $+3$ indicates that the decimal point must be moved to the right three places from its natural position and placed after 1. In problem (3) the characteristic being $+2$, the decimal point must be moved two places to the right of its natural position. In problem (4) 56490 is the mantissa for 3672, but since the characteristic is 5 the number must be between 10^5 and 10^6 or 367200. This is moving the decimal point five places to the right of the natural position it would have had for a zero characteristic. In each case the digits of the number are determined by finding the mantissa in the table, and the position of the decimal point by the characteristic. This process is called finding the antilogarithm.

In addition to their use in multiplication, logarithms can be used in division and in raising to powers and extracting roots. At least one lesson may be devoted to this feature for further drill in the use of logarithms before interpolation in logarithms and in antilogarithms is presented.

Since $a^5 / a^3 = a^{5-3} = a^2$, and since logarithms are exponents of ten, then to divide one number by another merely subtract the logarithms and find the number whose logarithm is that difference.

$$\begin{aligned} {}^{53380}\!/_{785} &= ? \\ \log \quad 785 &= 2.89487 \\ \hline \text{difference} &= 1.83251 = \log 68 \end{aligned}$$

Many problems in division with numbers of the size of those used for multiplication may be given. Problems can also be given involving raising to powers and extracting roots, but this can be done with less handicap after interpolation has been taught.

III. Interpolation.

Interpolation is not difficult in itself but when added to a lot of other new ideas it makes the whole seem unnecessarily difficult. Interpolation is necessary to find the logarithm of a number with five or more digits and also to find a number whose exact logarithm is not found in the tables. It can be most sim-

ply presented by the inductive method. Find the log of 66745. Only four digits can be found in the table.

$$\log 66740 = 4.82439$$
$$\log 66750 = 4.82445$$

It is impossible to find the logarithm of the number 66745 in the table but we can find the logarithm of two numbers, one a little smaller and one a little larger. Now since 66745 is just half way between 66740 and 66750, its logarithm will be half way between the logs of these two numbers, or log 63745 = 4.82442.

Have the class find the logarithms of several numbers whose fifth digit is 5, *e. g.*, 37.625, 498750, 267.75, etc. Then give them one whose fifth digit is 3 or 8 or some other number. Many will no doubt be able to interpolate with very little help.

$$\log 66743 = \log 66740 + \tfrac{3}{10} \text{ of } .00006$$
$$4.82439 + .000018 = 4.824408, \text{ or}$$
$$4.82441 \text{ to the nearest fifth place.}$$

$$\left.\begin{array}{l}\log 66740 = 4.82439 \\ \log 66743 = \\ \log 66750 = 4.82445\end{array}\right\} \text{difference between the logs is 6 in the fifth place.}$$

66743 is $\frac{3}{10}$ of the way from 66740 to 66750, therefore its logarithm must be $\frac{3}{10}$ of the way between these logarithms. Stated in another way, 66743 is greater than 66740 by $\frac{3}{10}$ of the difference between 66740 and 66750, therefore its logarithm must be greater than the logarithm of the smaller number by $\frac{3}{10}$ of the difference between the logarithms.

Let us summarize this process in what might be called a rule. To find the logarithm of a number with more than four digits find the logarithms of the numbers next larger and next smaller that can be found in the table. Find the difference betwen these logarithms and multiply this difference by the decimal fraction formed by putting a decimal point before the remaining digits. Add the product to the logarithm of the smaller number to get the logarithm of the desired number.

$$\log 66750 = 4.82445$$
$$\log 66740 = 4.82439$$

$$\begin{array}{rr} \text{difference} & .00006 \\ \text{multiply by} & .3 \\ \hline & .000018 \end{array}$$

add log 66740 = 4.82439

log 66743 = 4.824408 or
= 4.82441 to nearest fifth place.

Find the log 13736.

log 13740 = 4.16850
log 13730 = 4.16820

difference .00030
multiply by .6

.000180
add log 13730 = 4.16820

log 13736 = 4.16838

Find the log of 13738.45.

log 13740.00 = 4.16850
log 13730.00 = 4.16820

difference .00030
multiply by .845

.00025350
add log 13730.00 = 4.16820

log 13738.45 = 4.16845 to nearest fifth place.

There is a second problem in interpolation, and that is to find the number when the exact mantissa cannot be found in the table. Find the number whose logarithm is 4.33776. The mantissas 33786 and 33766 can be found in the table. It is evident then that 33776 must be half way between these two.

log 21770 = 4.33786
log 21760 = 4.33766

difference 10 .00020
log = 4.33776
log 21760 = 4.33766

.00010
.00010 / .00020 = .5

Therefore the number is $21760 + .5 \times 10 = 21765$.

Find the number whose log is 2.33781.

$$\begin{aligned} \log\ 217.7 &= 2.33786 \\ \log\ 217.6 &= 2.33766 \\ \text{difference} &= .00020 \\ \log\ &= 2.33781 \\ \log\ 217.6 &= 2.33766 \\ \text{difference} &= .00015 \end{aligned}$$

$$\text{Therefore the number} = 217.6 + \frac{.00015}{.00020} \times .1 = 217.675.$$

Since the difference between the numbers will always be 10, 100, .1, or powers of ten, the correct digits will always be found by dividing the smaller difference by the larger: $10 \div 20 = .5$ and $15 \div 20 = .75$. In the first problem 5 became the fifth digit, and in the second 7 the fifth and 5 the sixth. The decimal point in the final number will be determined by the characteristic. It will be moved as many places to the right or left of its natural position as the characteristic indicates.

Find number whose logarithm is 3.46364.

$$\begin{aligned} \log\ 2909 &= 3.46374 \\ \log\ 2908 &= 3.46359 \\ \text{difference} &\quad .00015 \\ \log\ &= 3.46364 \\ \log\ 2908 &= 3.46359 \\ \text{difference} &\quad .00005 \\ 5\ /\ 15 &= .3333 \end{aligned}$$

Therefore the number equals 2908.333.

Find number whose logarithm is 2.46360.

$$\begin{aligned} \log\ &= 2.46360 \\ \log\ 290.8 &= 2.46359 \\ \text{difference} &\quad .00001 \\ \log\ 290.9 &= 2.46374 \\ \log\ 290.8 &= 2.46359 \\ \text{difference} &\quad .00015 \\ .0001\ /\ .00015 &= 1\ /\ 15 = .066 \end{aligned}$$

Therefore the number is 290.8066+.

Let us also summarize this process in what may be called a rule. To find the number when the given logarithm cannot be found directly in the table, find the numbers whose logs are the next smaller and the next larger in the table. *Find two differences,* the difference between the two logs from the table and the difference between the given logarithm and the smaller of the other two. *Divide the smaller difference by the larger.* Place the digits obtained at the right of the smaller number and place the decimal point as indicated by the characteristic.

Now the student is ready for any or all problems except those with numbers less than one. He should be given drill on the use of the knowledge acquired. This should include multiplication and division, extracting roots and raising to powers.

Find the square root of 4761.

$$\log 4761 = 3.67770$$
$$\text{divide by } 2 = 1.83885 = \log 69$$

Find the cube root of 500.

$$\log 500 = 2.69897$$
$$\text{divide by } 3 + 0.89966 = \log 7.9370$$

Find $(37)^3$.

$$\log 37 = 1.56820$$
$$\text{multiply by } 3 = 4.70460 = \log 50653$$

To extract a root divide the logarithm of the number by the indicated root, to raise a number to a power multiply the logarithm by the power, and then in each case look up the number having this result for its logarithm. The $\sqrt[3]{a^{12}} = a^{12/3} = a^4$, and $(a^2)^3 = a^{3 \cdot 2} = a^6$, are of course the foundation for the procedure with the logarithms.

The term cologarithm may be introduced with the best classes. It is not essential for a complete understanding of logarithms, however. This term and the solution of exponential equations are simple to present and will be omitted from this paper. There remains then only the use of logarithms with numbers less than one such as .0964.

The only difference between this and the work already presented is the negative characteristic.

Find log .0964.

$$\log .0964 = -2 + .98408 = 8.98408 - 10.$$

Since the decimal point is 2 places to the left of its natural position the characteristic is -2. However, rather than have the negative logarithm corresponding to a negative exponent, 10 is added to the -2 and subtracted again after the mantissa. Thus $-2 + .98408 = -2 + 10.98408 - 10 = 8.98408 - 10$.

Multiply .0964 by 8250.

$$\log .0964 = 8.98408 - 10$$
$$\log 8250 = 3.91645$$
$$\overline{\qquad\qquad\qquad}$$
$$12.90053 - 10 = 2.90053 = \log 795.3$$

The logarithm of the decimal less than one can be used just as any other logarithm is used. Care should be taken to carry along the minus ten and to subtract and drop it when possible.

In extracting a root of a decimal fraction less than one, which involves dividing the logarithm by the indicated root, the teacher may experience some difficulty.

Find cube root of .042875.

$$\log .042875 = 8.63220 - 10$$
$$\text{divide by } 3 \quad 2.87740 - 3\tfrac{1}{3}$$

But $3\frac{1}{3}$ is not an integer and therefore when subtracted from 2.87740 will affect the mantissa as well as the characteristic. Therefore to avoid this difficulty enough more is subtracted from ten to make the quotient integral, and the same amount added to the characteristic.

$$\log .042875 = 8.63220 - 10$$
$$+20 \qquad -20$$
$$\overline{\qquad\qquad\qquad}$$
$$= 28.63220 - 30$$

Divide by $3 - 9.54407 - 10 = \log .35$. The same result would be obtained the other way eventually by subtracting the one-third. $2.87740 - 3.33333 = 2.54407 - 3 = 9.54407 - 10 = \log .35$. The suggested process is more simple, however.

The aim of this paper has been to present the teaching of logarithms based on sound psychology of learning and on experience in analyzing pupils' difficulties. Brevity has been sacrificed to complete rationalization.

Plane Trigonometry. By Paul R. Rider and Alfred Davis. D Van Nostrand Company. New York, 1923, Pp. 280.

The text includes ten chapters, as follows: Introduction, Angles, The Trigonometric Functions, Logarithms, Right Triangles, Formulas and Identities, Oblique Triangles, Graphic Representation of Functions, Inverse Functions and Equations, and Analytic Trigonometry. Thus it is a "complete course." The Introduction contains important principles of algebra and theorems from geometry, a discussion of the place of trigonometry in the mathematical subjects, and notes on how to study. Students will be impressed with the discussion of the "importance of trigonometry" which states:

"Some of the important results to be gained from the study of other branches of mathematics may be realized in the study of trigonometry. It should develop the ability to think clearly and accurately. It may be made to develop accuracy, rapidity, and neatness; accuracy because this is essential in all mathematics; rapidity because it provides easy and short roads to conclusions and results; neatness—without which other advantages are frequently turned to naught—because of the orderly form required in the arrangement of the solutions of problems. But by far its greatest value consists in its practical application to problems that one meets on every hand. Indeed, it is one of the most practical branches of mathematics, and, with the proper background of algebra and geometry, it should be an unusually interesting subject to the student. It should serve as a fitting climax to a high school course in mathematics, and as a suitable beginning for a college course.

"As has been stated, trigonometry had its origin in astronomy, and one could know little of that science were it not for the trigonometric functions. By means of them we can reach to other suns, and learn of their stupendous magnitudes and velocities; they assist us in tracing out the paths that these heavenly bodies have traversed through the centuries and in predicting their movements for ages to come."

Among the directions for study are:

1. Read carefully the assigned lesson. Be sure that every word is understood and that the principles are clearly grasped. It frequently happens that much of the difficulty of the student

is due to careless reading or to guessing the meanings of words.

2. Connect the ideas involved with other facts already known. Try to justify mentally each point in the argument by calling up reasons.

3. Try to ascertain the main thought or the fundamental idea. This will aid in the untangling of complicated parts.

4. Express the newly acquired information in your own words. It is only as we are able to give expression to ideas that we are really master of them. Better still, endeavor to apply the new principle learned in some practical way. This will help to fix it in your mind.

5. If there is any point in the lesson which is not clear, try to formulate your difficulty and put it in writing. Try again to understand the matter, and if you fail present it to the class or to the teacher at the earliest opportunity.

6. In attempting to understand a difficult lesson use every mental resource you can command. For further help, consult other parts of the text through the table of contents and the index, refer to other texts, etc. If still in difficulty, discuss the matter with someone who can help you.

7. Be determined to master the lesson. A half-hearted attitude is almost certain to end in defeat. Do not discount, or allow anyone to discount for you, the importance of putting your best effort into the task you have undertaken.

8. Take time to study the lesson. The mental operations of real study require time as well as concentrated effort.

9. Remember in working problems that accuracy is more important than speed. Strive for accuracy; speed will come with practice.

10. Form the habit of self-reliance. Check your work to test its accuracy, and as a rule avoid answer books.

11. Above all, be sure to gain a thorough mastery of each point as it comes. Do not allow anything to pass not understood or poorly understood. Failure here is certain to result in discouragement and ultimately in defeat. Do not depend on getting the matter later or on "cramming" before an examination.

The reviewer notes the generous use of graphic methods, simplicity of devices for computation, and a large number of excellent problems and applications.